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Thomas B. Woodward

AN

INTRODUCTION

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TO

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ALGEBRA;

WITH

NOTES AND OBSERVATIONS;

DESIGNED

FOR THE USE OF SCHOOLS AND PLACES
OF PUBLIC EDUCATION.

BY JOHN BONNYCASTLE, 1750? – 1821

OF THE ROYAL MILITARY ACADEMY, WOOLWICH.

The First American Edition.

——— Ingenuas didicisse fideliter artes
Emollit mores, nec sinit esse feros. OVID.

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PREFACE.

THE powers of the mind, like those of the body, are increased by frequent exertion; application and industry supply the place of genius and invention; and even the creative faculty itself may be strengthened and improved by use and perseverance. Uncultivated nature is uniformly rude and imbecile, it being by imitation alone that we at first acquire knowledge, and the means of extending its bounds. A just and perfect acquaintance with the simple elements of science, is a necessary step towards our future progress and advancement; and this, assisted by laborious investigation and habitual inquiry, will constantly lead to eminence and perfection.

Books of rudiments, therefore, concisely written, well digested, and methodically arranged, are treasures of inestimable value; and too many attempts cannot be made to render them perfect and complete. When the first principles of any art or science are firmly fixed and rooted in the mind, their application soon becomes easy, pleasant, and obvious; the understanding is delighted and enlarged; we conceive clearly, reason distinctly, and form just and satisfactory conclusions. But, on the contrary, when the mind, instead of reposing on the stability of truth and received principles, is wandering in doubt and uncertainty, our ideas will necessarily be confused and obscure; and every step we take must be attended with fresh difficulties and endless perplexity.

That the grounds, or fundamental parts, of every science, are dull and unentertaining, is a complaint universally made, and a truth not to be denied; but, then, what is obtained with difficulty is usually remem

bered with ease; and what is purchased with pain is possessed with pleasure. The seeds of knowledge are sown in every soil, but it is by proper culture alone that they are cherished and brought to maturity. A few years of early and assiduous application never fails to procure us the reward of our industry; and who is there, who knows the pleasures and advantages which the sciences afford, that would think his time mis-pent, or his labours useless? Riches and honours are the gifts of fortune, casually bestowed, or hereditarily received, and are frequently abused by their possessors; but the superiority of wisdom and knowledge is a pre-eminence of merit, which originates with the man, and is the noblest of all distinctions.

Nature, bountiful and wise in all things, has provided us with an infinite variety of scenes, both for our instruction and entertainment; and, like a kind and indulgent parent, admits all her children to an equal participation of her blessings. But, as the modes, situations, and circumstances of life are various, so accident, habit, and education, have each their predominating influence, and give to every mind its particular bias. Where examples of excellence are wanting, the attempts to attain it are but few; but eminence excites attention, and produces imitation. To raise the curiosity, and to awaken the listless and dormant powers of younger minds, we have only to point out to them a valuable acquisition, and the means of obtaining it. The active principles are immediately put into motion, and the certainty of the conquest is ensured from a determination to conquer.

But, of all the sciences which serve to call forth this spirit of enterprise and inquiry, there are none more eminently useful than the Mathematics. By an early attachment to these elegant and sublime studies, we acquire a habit of reasoning, and an elevation of thought, which fixes the mind, and prepares it for every other pursuit. From a few simple axioms, and

evident principles, we proceed gradually to the most general propositions, and remote analogies: deducing one truth from another, in a chain of argument well connected and logically pursued; which brings us at last in the most satisfactory manner, to the conclusion, and serves as a general direction in all our inquiries after truth.

And it is not only in this respect that mathematical learning is so highly valuable; it is, likewise, equally estimable for its practical utility. Almost all the works of art, and devices of man, have a dependence upon its principles, and are indebted to it for their origin and perfection. The cultivation of these admirable sciences is, therefore, a thing of the utmost importance, and ought to be considered as a principal part of every liberal and well-regulated plan of education. They are the guide of our youth, the perfection of our reason, and the foundation of every great and noble undertaking.

From these considerations, I have been induced to undertake an introductory course of mathematical science; and, from the kind encouragement which I have hitherto received, am not without hopes of a continuance of the same candour and approbation. Considerable practice as a teacher, and a long attention to the difficulties and obstructions which retard the progress of learners in general, have enabled me to accommodate myself the more easily to their capacities and understandings. And as an earnest desire of promoting and diffusing useful knowledge is the chief motive for this undertaking, so no pains or attention shall be wanting to make it as complete and perfect as possible.

The subject of the present performance is ALGEBRA; which is one of the most important and useful branches of those sciences, and may be justly considered as the key to all the rest. Geometry delights us by the simplicity of its principles, and the elegance of its

demonstrations. Arithmetic is confined in its object, and partial in its application: but Algebra, or the analytic art, is general and comprehensive, and may be applied with success in all cases where truth is to be obtained and proper data can be established.

To trace this science to its birth, and to point out the various alterations and improvements it has undergone in its progress, would far exceed the limits of a preface. It will be sufficient to observe that the invention is of the highest antiquity, and has challenged the praise and admiration of all ages. *Diophantus* appears to have been the first, among the ancients, who applied it to the solution of indeterminate or unlimited problems; but it is to the moderns that we are principally indebted for the most curious refinements of the art, and its great and extensive usefulness in every abstruse and difficult inquiry. *Newton Maclaurin*, *Sanderson*, *Simpson*, and *Emerson*, are those of our own countrymen, who have particularly excelled in this respect; and it is to their works that I would refer the young student, as the patterns of elegance and perfection.

The following compendium is formed entirely upon the model of those writers, and is intended as an useful and necessary introduction to them. Almost every subject, which belongs to pure Algebra, is concisely and distinctly treated of: and no pains have been spared to make the whole as easy and intelligible as possible. A great number of elementary books have already been written upon this subject; but there are none, which I have yet seen, but what appear to me to be extremely defective. Besides being totally unfit for the purpose of teaching, they are generally calculated to vitiate the taste, and mislead the judgment. A tedious and inelegant method prevails through the whole, so that the beauty of the science is generally destroyed by the clumsy and awkward manner in which it is treated; and the learner, when he is afterwards

introduced to some of our best writers, is obliged to unlearn and forget every thing which he has been at so much pains in acquiring.

It is in the sciences as in every branch of polite literature; there is a certain taste and elegance which is only to be obtained from the best authors, and a judicious use of their instructions. To direct the student in his choice of books, and to prepare him properly for the advantages he may receive from them, is, therefore, the business of every writer who engages in the humble, but useful task of a preliminary tutor. This information I have been careful to give, in every part of the present performance, where it appeared to be in the least necessary; and, though the nature and confined limits of my plan admitted not of diffuse observations, or a formal enumeration of particulars, it is presumed nothing of real use and importance has been omitted. My principal object was to consult the ease, satisfaction, and accommodation of the learner; and the favourable reception the work has met with from the public, has induced me to give this edition an attentive and careful revisal.

John Burr
Burlington

THE

CONTENTS.

ALGEBRA.

DEFINITIONS.

ALGEBRA is the art of computing by ſymbols.

1. *Like quantities* are thoſe which conſiſt of the ſame letters.

2. *Unlike quantities* are thoſe which conſiſt of different letters.

3. *Given quantities* are thoſe whoſe values are known.

4. *Unknown quantities* are thoſe whoſe values are unknown.

5. *Simple quantities* are thoſe which conſiſt of one term only.

6. *Compound quantities* are thoſe which conſiſt of ſeveral terms.

7. *Poſitive* or *affirmative quantities* are thoſe which are to be added.

8. *Negative quantities* are thoſe which are to be ſubtracted.

9. *Like ſigns* are all affirmative (+), or all negative (—).

10. *Unlike ſigns* are when ſome are affirmative (+) and others negative (—).

11. *The co-efficient* of any quantity, is the number prefixed to it.

12. *A binomial quantity*, is one conſiſting of two terms; a *trinomial* of three terms; a *quadrinomial* of four, &c.

13. *A reſidual quantity* is a binomial where one of the terms is negative.

14. *The power of a quantity* is its ſquare, cube, biquadrate, &c.

15. *The index* or *exponent of a quantity* is the number which denotes its root or power.

16. *A surd* or *irrational quantity*, is that which has no exact root.

17. *A rational quantity* is that which has no radical ſign ($\surd$) or index annexed to it.

18. *The reciprocal* of any quantity is that quantity inverted, or unity divided by it.

EXPLANATION OF THE CHARACTERS.

$+$	Is the ſign of	addition.
$-$	———	of ſubtraction.
$\times$	———	of multiplication.
$\div$	———	of diviſion,
$:$	$::$ $:$	of proportion.
$\surd$	———	of the ſquare root.
$\surd$	———	of the cube root.
$=$	———	of equality.

Thus, $a+b$ ſhews that the number repreſented by b is to be added to that repreſented by a.

$a-b$ ſhews that the number repreſented by b is to be ſubtracted from that repreſented by a.

$a \backsim b$ repreſents the difference of a and b when it is not known which is the greateſt.

ab, or $a\times b$, or $a.b$, denotes the product of the numbers repreſented by a and b.

$a \div b$, or $\frac{a}{b}$, ſhews that the number repreſented by a is to be divided by that repreſented by b.

$a : b :: c : d$ denotes that a is in the ſame proportion to b as c is to d.

$x = a - b + c$ is an equation, ſhewing that x is equal to the difference of a and b, added to the quantity c.

$\sqrt{a}$, or $a^{\frac{1}{2}}$, is the ſquare root of a; $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, is the cube root of a; and $a^{\frac{1}{n}}$ is the nth root of a.

a^2 is the ſquare of a; a^3 the cube of a; a^4 the fourth power of a; and a^m the mth power of a.

$\frac{a}{b}$ is the reciprocal of $\frac{b}{a}$, and $\frac{1}{a}$ the reciprocal of a.

$\overline{a+b} \times c$, or $(a+b)c$ is the product of the compound quantity $a+b$ multiplied by the ſimple quantity c.

$\overline{a+b} \div \overline{a-b}$, or $(a+b) \div (a-b)$, or $\frac{a+b}{a-b}$; is the quotient of $a+b$ divided by $a-b$.

$\sqrt{ab+cd}$ or $(ab+cd)^{\frac{1}{2}}$ is the ſquare root of the compound quantity $ab+cd$.

$\overline{a+b-c}^3$ or $(a+b-c)^3$ is the cube, or third power, of the quantity $a+b-c$.

$5a$ denotes that the quantity a is to be taken 5 times, and $7(b+c)$, is 7 times $b+c$

It is alſo to be remarked that the ſign $+$ is generally expreſſed by the word *plus*, or *more*, and the ſign $-$, by *minus*, or *less*.

And in the computation, of problems, it muſt be obſerved, that the firſt letters of the alphabet are uſually put for known quantities, and the laſt for thoſe which are unknown.

ADDITION.

CASE I.

To add quantities which are like, and have like signs.

RULE.

Add all the co-efficients together, and to their sum adjoin the letters common to each term, prefixing the common sign.

EXAMPLES.*

$5\,a$	$-6\,bx$	$8\,bxy$
$7\,a$	$-3\,bx$	$7\,bxy$
$8\,a$	$-2\,bx$	$3\,bxy$
$10\,a$	$-7\,bx$	$4\,bxy$
$2\,a$	$-\,bx$	$5\,bxy$
a	$-5\,bx$	bxy
$33\,a$	$-24\,bx$	$28\,bxy$

$2y$	$5x^2+5xy$	$7ax-4y$
$5y$	$3x^2+2xy$	$8ax-3y$
$7y$	x^2+3xy	$6ax-2y$
$4y$	$7x^2+8xy$	$4ax-3y$
$3y$	$x^2+\,xy$	$2ax-2y$
$21y$	$17x^2+19xy$	$27ax-14y$

* When a leading quantity has no sign before it, + is always understood; and a quantity without any co-efficient prefixed to it, is supposed to have 1 or unity.

$6\,xy$	$-2y^2$	$5a-4b$
$15\,xy$	$-8y^2$	$7a-6b$
$2\,xy$	$-7y^2$	$4a-3b$
$7\,xy$	$-\ y^2$	$2a-8b$
xy	$-6y^2$	$6a-\ b$
xy	$-\ y^2$	$3a-2b$

* $20-15x^{\frac{1}{2}}-2\,xy$	$7\,xy-5x+3\,ab$
$35-13x^{\frac{1}{2}}-4\,xy$	$3\,xy-\ x+2\,ab$
$18-12x^{\frac{1}{2}}-3\,xy$	$2\,xy-3x+2\,ab$
$12-14x^{\frac{1}{2}}-8\,xy$	$2\,xy-4x+8\,ab$
$10-28x^{\frac{1}{2}}-2\,xy$	$5\,xy-3x+\ ab$

CASE II.

To add quantities which are like, but have unlike signs.

RULE.

1. Add all the affirmative co-efficients into one ſum, and all the negative ones into another.

2. Subtract the leaſt of theſe ſums from the greateſt, and to the difference prefix the ſign of the greateſt, with the common quantity.

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* Quantities, having fractional or other exponents, are to be conſidered, in all reſpects, as if they were expreſſed by a ſingle letter.

EXAMPLES.

$$\begin{array}{r}-3a\\+7a\\+8a\\-a\\-2a\\\hline+9a\\\hline\end{array}\qquad\begin{array}{r}+8ax^2\\+7ax^2\\-3ax^2\\-4ax^2\\+4ax^2\\\hline+12ax^2\\\hline\end{array}\qquad\begin{array}{r}+6x^3+8y\\-3x^3+7y\\-13x^3+8y\\+2x^3-3y\\+x^3-y\\\hline-7x^3+19y\\\hline\end{array}$$

$$\begin{array}{r}-2a^2\\-3a^2\\-8a^2\\+10a\\+13a^2\\\hline\\\hline\end{array}\qquad\begin{array}{r}+8b^2y^3\\+6b^2y^3\\-10b^2y^3\\-20b^2y^3\\-b^2y^3\\\hline\\\hline\end{array}\qquad\begin{array}{r}-3ab+7\\+3ab-10\\+3ab-6\\-ab+2\\-2ab+11\\\hline\\\hline\end{array}$$

$$\begin{array}{r}-5xy\\-3xy\\+8xy\\+7xy\\\hline\\\hline\end{array}\qquad\begin{array}{r}-8x^2y^2\\+3x^2y^2\\-2x^2y^2\\+4x^2y^2\\\hline\\\hline\end{array}\qquad\begin{array}{r}-12x^2-8x\\+10x^2-3x\\-14x^2+7x\\+8x^2+x\\\hline\\\hline\end{array}$$

$$\begin{array}{r}-2ax^{\frac{1}{2}}\\+ax^{\frac{1}{2}}\\-3ax^{\frac{1}{2}}\\+7ax^{\frac{1}{2}}\\\hline\\\hline\end{array}\qquad\begin{array}{r}-6\sqrt{ax}\\+2\sqrt{ax}\\-6\sqrt{ax}\\+10\sqrt{ax}\\\hline\\\hline\end{array}\qquad\begin{array}{r}-2y+2ax^{\frac{1}{2}}\\+y+ax^{\frac{1}{2}}\\-7y-3ax^{\frac{1}{2}}\\+5y+3ax^{\frac{1}{2}}\\\hline\\\hline\end{array}$$

CASE III.

To add quantities which are unlike and have unlike signs.

RULE.

Collect all the like quantities together by the last rule, and set down those that are unlike, one after another, with their proper signs.

EXAMPLES.

$5xy$	$2xy-10x^2$	$2ax-150+2x^{\frac{1}{2}}$
$4ax$	$-3x^2+xy$	$3x^2+2ax+6x^2$
$-xy$	$-8x^2-xy$	$4xy-3x^{\frac{1}{2}}+50$
$-4ax$	$-xy+9x^2$	$\sqrt{x}+100-5x^2$
$4xy$	$xy-12x^2$	$4ax+4x^2+4xy$

$6x^2y^2$	$12xa-x^2$	$6+20\sqrt{ax}-3y$
$-4x^2y$	$4ax+xy$	$x+4\sqrt{xy}+3y$
$-2axy$	$3y^2-ax$	$y-2\sqrt{ax}-3y$
$-3x^2y$	$2x^2-24$	$20+3\sqrt{ax}-3y$

$3x^2y$	$2\sqrt{x}-8y$	$a^2-8+x^{\frac{1}{2}}-2$
$-2xy^2$	$3\sqrt{xy}+10x$	$a-10+a^2-x$
$-3y^2x$	$2x+\sqrt{x+y}$	x^2-a^2+8-4
$-8x^2y$	$-8+\sqrt{xy}$	$10-a-x^2-y$

SUBTRACTION.

RULE.

Change the signs of all the quantities to be subtracted, or conceive them to be changed, and then collect the different terms together as in addition.

EXAMPLES.

$5a^2-2b$	$6x^2-8y+3$	$5xy-2+8x-y$
$2a^2-5b$	$2x^2+9y-2$	$3xy-8-8x-3y$
$3a^2+3b$	$4x^2-17y+5$	$2xy+6+16x+3y$

$3xy-8$	$2y^2-y-1$	$-10-8x-3xy$
$-xy+8$	y^2+y+1	$xy-7x+3-4ay$
$4xy-16$	y^2-2y-2	$-13-x-4xy+4ay.$

$5x^2y-8$	$4\sqrt{xy}-x\sqrt{xy}$	$5x^2+\sqrt{x}-8-4b^{\frac{1}{2}}$
$-3x^2y+1$	$2\sqrt{xy}+2+xy$	$6x^2-10+4b-x^{\frac{1}{2}}$

$3xy-20$	$4x^3-3.(a+b)$	$xy^3+10a\sqrt{(xy+10)}$
$4xy-30$	$3x^3-8.(a+b)$	$x^2y^2+2a\sqrt{(xy+10)}$

MULTIPLICATION.

CASE I.

When both the factors are simple quantities.

RULE.

Multiply the co-efficients of the two terms together, and to the product affix all the letters in those terms, and the result will be the whole product required.

*Note**. Like signs produce +, and unlike signs —.

EXAMPLES.

$12a$	$-2a$	$5a$	$-9x$
$3b$	$+4b$	$-6x$	$-5b$
$36ab$	$-8ab$	$-30ax$	$+45bx$
$7ab$	$6a^2x$	$-x^2y$	$-7xy$
$-5ac$	$5x$	xy^2	$-xy$
$-35a^2bc$	$30a^2x^2$	x^3y^3	$+7x^2x^2$
$-5ax$	$-ax$	$+5xy$	$-7xyx$
$3x$	$-7b$	-3	$-6ax$
$8ax^2$	$5x^2y^2$	$-3xy^2$	$17x^2y^2$
$3ax$	x^2y^2	$+2ax$	$2ax^2$

* That like signs make +, and unlike signs —, in the product, may be shewn thus:

1st. When $+a$ is to be multiplied by $+b$: this implies that $+a$ is

CASE II.

When one of the factors is a compound quantity.

RULE.

Find the products of the multiplier and every term of the multiplicand, separately, and place them one after another, with the proper signs, and the result will be the whole product required.

EXAMPLES.

$4a-2b$	$6xy-8$	a^2-2x+6
$3a$	$2x$	xy
$12a^2-6ab$	$12x^2y-16x$	a^2xy-2x^2y+6xy

$13x-ab$	$35x-7a$	$3y-8+2xy$
$12a$	$-x$	xy

$2x^2+ay$	$12x^2-4y^2$	$2y^2-8x^2-7x$
$2xy$	$-2x^2$	$3xy^2$

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is to be taken as many times as there are units in b; and, since the sum of any number of affirmative terms is affirmative, it is plain that $(+a) \times (+b) = +ab$.

2. If two quantities are to be multiplied together, the result will be exactly the same, in whatever order they are placed; for a times b is the same as b times a: and, therefore, when

CASE III.

When both the factors are compound quantities.

RULE.

Multiply every term of the multiplier into every term of the multiplicand respectively, and set down the products one after another with their proper signs, and their sum will be the whole product required.

$$\begin{array}{l} x+y \\ x+y \\ \hline x^2+xy \\ \quad +xy+y^2 \\ \hline x^2+2xy+y^2 \\ \hline \end{array} \qquad \begin{array}{l} 5x+4y \\ 3x-2y \\ \hline 15x^2+12xy \\ \quad -10xy-8y^2 \\ \hline 15x^2+2xy-8y^2 \\ \hline \end{array} \qquad \begin{array}{l} x^2+xy-y^2 \\ \quad x-y \\ \hline x^3+x^2y-xy^2 \\ \quad -x^2y-xy^2+y^3 \\ \hline x^3 \; * -2xy^2+y^3 \\ \hline \end{array}$$

$$\begin{array}{l} x+y \\ x-y \\ \hline x^2+xy \\ \quad -xy-y^2 \\ \hline x^2 \; * -y^2 \\ \hline \end{array} \qquad \begin{array}{l} x^2+y \\ x^2+y \\ \hline x^4+yx^2 \\ \quad +yx^2+y^2 \\ \hline x^4+2yx^2+y^2 \\ \hline \end{array} \qquad \begin{array}{l} x^2+xy+y^2 \\ x-y \\ \hline x^3+x^2y+xy^2 \\ \quad -x^2y-xy^2-y^3 \\ \hline x^3 \; * \; * \; -y^3 \\ \hline \end{array}$$

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when $-a$ is to be multiplied by $+b$, or $+b$ by $-a$, this is the same as taking $-a$ as many times as there are units in $+b$; and since the sum of any number of negative terms is negative, it is evident that $(-a)\times(+b)$ or $(+b)\times(-a)=-ab$.

3. When $-a$ is to be multiplied by $-b$: Here $a-a=o$, therefore $(a-a)\times -b$ is also $=o$, because o multiplied by any quantity produces o; and since the first term of the product, or $a\times(-b)$, is, by case 2, $=-ab$, the last term, or $(-a)\times(-b)$, must be $=+ab$, in order to make the sum $(-ab+ab)=o$ consequently $(-a)\times(-b)=+ab$.

EXAMPLES FOR PRACTICE.

1. Multiply $12ax$ by $3a$. *Anf.* $36a^2x$.
2. Multiply $4x^2-2y$ by $2y$. *Anf.* $8x^2y-4y^2$.
3. Multiply $2x+4y$ by $2x-4y$. *Anf.* $4x^2-16y^2$.
4. Multiply x^2-xy+y^2 by $x+y$. *Anf.* x^3+y^3.
5. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. *Anf.* x^4-y^4.
6. Multiply x^2+xy+y^2 by x^2-xy+y^2.
7. Multiply $3x^2-2xy+5$ by $x^2+2xy-3$
8. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$
9. Multiply $3x^3+2x^2y^2+3y^3$ by $2x^3-3x^2y^2+5y^3$.

DIVISION.

CASE I.

When the divifor is a fimple quantity.

RULE.

1. Place the dividend above a fmall right line, and the divifor under it, in the manner of a vulgar fraction.

2. Expunge thofe letters which are common to both the dividend and divifor, and divide the co-efficients of all the terms by any number that will divide them without a remainder, and the refult will be the quotient required.

*Note**. Like figns make +, and unlike figns —, the fame as in multiplication.

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* That like figns give +, and unlike figns — in the quotient will appear thus:

The divifor multiplied by the quotient muft produce the dividend; therefore

1. When

EXAMPLES.

1. It is required to find the quotient of $a \div a$; $8bc \div 2b$; and $abc \div bcd$.

Ans. $\frac{a}{a}=1$; $\frac{8bc}{2b}=4c$; and $\frac{abc}{bcd}=\frac{a}{d}$

2. It is required to find the quotient of $12xy \div 6x^2$; and $(ab+b^2) \div 2b$

Ans. $\frac{12xy}{6x^2}=\frac{2y}{x}$, and $\frac{ab+b^2}{2b}=\frac{a+b}{2}$

3. Divide $18x^2$ by $9x$. *Ans.* $2x$.
4. Divide $10x^2y^2$ by $-5x^2y$. *Ans.* $-2y$.
5. Divide $-9ax^2y^2$ by $9x^2y$. *Ans.* $-ay$.
6. Divide $-8x^2$ by $-2x$. *Ans.* $+4x$.
7. Divide $10\,ab+15\,ac$ by $5a$. *Ans.* $2b+3c$.
8. Divide $30ax-54x$ by $6x$ *Ans.* $5a-9$.
9. Divide $10x^2y-15y^2-5y$ by $5y$. *Ans.* $2x^2-3y-1$.
10. Divide $13a+3ax-17x^2$ by $21a$.
11. Divide $3a^2-15+6a+3b$ by $3a$.

CASE II.

When the divisor and dividend are both compound quantities.

RULE.

1. Range the terms of both the quantities according to the dimensions of some letter in them, so that the

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1. When both the terms are +, the quotient is + because $(+)\times(+)$ produces more in the dividend.
2. When they are both —, the quotient is also + because $(+)\times(-)$ produces — in the dividend.
3. When one of them is + and the other — the quotient is —, because $(-)\times(+)$ produces — in the dividend; and $(-)\times(-)$ produces + in the dividend.

the firſt term may contain the higheſt power of that letter, the ſecond term, the next higheſt power; and ſo on.

2. Divide the firſt term of the dividend by the firſt term of the diviſor, and place the reſult in the quotient.

3. Multiply the whole diviſor by the term thus found, and ſubtract the reſult from the dividend.

4. To this remainder bring down as many terms of the dividend as are requiſite for the next operation, and divide as before; and ſo on, as in common arithmetic.

Note. If the diviſor be not exactly contained in the dividend, the quantity which remains after the operation is finiſhed, muſt be placed over the diviſor, like a vulgar fraction, and ſet down at the end of the quotient, as in common arithmetic.

EXAMPLES.

$$x+y)x^2+2xy+y^2(x+y$$
$$x^2+xy$$
$$\overline{}$$
$$xy+y^2$$
$$xy+y^2$$
$$\overline{}$$
$$*$$

$$a+x)a^3+5a^2x+5ax^2+x^3(a^2+4ax+x^2$$
$$a^3+a^2x$$
$$\overline{}$$
$$4a^2x+5ax^2$$
$$4a^2x+4ax^2$$
$$\overline{}$$
$$ax^2+x^3$$
$$ax^2+x^3$$
$$\overline{}$$
$$*$$

$$x-3)\,x^3-9x^2+27x-27\,(x^2-6x+9$$
$$x^3-3x^2$$
$$\overline{\quad -6x^2+27x}$$
$$-6x^2+18x$$
$$\overline{\quad 9x-27}$$
$$9x-27$$
$$\overline{\quad *\quad}$$

$$a-x)\,a^3-x^3\,(a^2+ax+x^2$$
$$a^3-a^2x$$
$$\overline{\quad a^2x-x^3}$$
$$a^2x-ax^2$$
$$\overline{\quad ax^2-x^3}$$
$$ax^2-x^3$$
$$\overline{\quad *\quad}$$

$$b-y)\,b^4-y^4\,(b^3+b^2y+by^2+y^3$$
$$b^4-b^3y$$
$$\overline{\quad b^3y-y^4}$$
$$b^3y-b^2y^2$$
$$\overline{\quad b^2y^2-y^4}$$
$$b^2y^2-by^3$$
$$\overline{\quad by^3-y^4}$$
$$by^3-y^4$$
$$\overline{\quad *\quad}$$

EXAMPLES FOR PRACTICE.

1. Divide $a^2+2ax+x^2$ by $a+x$. *Anf.* $a+x$.
2. Divide $a^3-3a^2y+3ay^2-y^3$ by $a-y$. *Anf.* $a^2-2ay+y^2$.
3. Divide 1 by $1-x$ *Anf.* $1+x+x^2+x^3$, &c.
4. Divide $6x^4-96$ by $3x-6$ *Anf.* $2x^3+4x^2+8x+16$.
5. Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$. *Anf.* $a^3-3a^2x+3ax^2-x^3$.
6. Divide $48x^3-76ax^2-64a^2x+105a^3$ by $2x-3a$.
7. Divide $y^6-3y^4x^2+3y^2x^4-x^6$ by $y^3-3y^2x+3yx^2-x^3$.

ALGEBRAIC FRACTIONS.

PROBLEM I.

To reduce a mixed quantity to an improper fraction.

RULE.

Multiply the integer by the denominator of the fraction, and to the product add the numerator; and the denominator being placed under this fum, will give the improper fraction required.

EXAMPLES.

1. Let $3\frac{5}{7}$, and $a-\frac{b}{c}$ be reduced to improper fractions.

Firft, $3\frac{5}{7}=\frac{3\times 7+5}{7}=\frac{21+5}{7}=\frac{26}{7}$ *Anf.*

And, $a-\frac{b}{c}=\frac{a\times c-b}{c}=\frac{ac-b}{c}$. *Anf.*

2. Let $x+\frac{x^2}{a}$ and $x-\frac{a^2-x^2}{x}$ be reduced to improper fractions.

First, $x+\frac{x^2}{a}=\frac{x\times a+x^2}{a}=\frac{ax+x^2}{a}$ *the Answer.*

And $x-\frac{a^2-x^2}{x}=\frac{x^2-a^2+x^2}{x}=\frac{2x^2-a^2}{x}$ *Ans.*

3. Reduce $8\frac{6}{7}$ to an improper fraction. *Ans.* $\frac{62}{7}$

4. Reduce $1-\frac{2x}{a}$ to an improper fraction. *Ans.* $\frac{a-2x}{a}$

5. Let $x-\frac{ax+x^2}{2a}$ be reduced to an improper fraction.

6. Let $10+\frac{2x-8}{3x}$ be reduced to an improper fraction.

7. Let $a+\frac{1-x-b}{b}$ be reduced to an improper fraction.

8. Let $1+2x-\frac{x-3}{5x}$ be reduced to an improper fraction.

PROBLEM II.

To reduce an improper fraction to a whole or mixed quantity.

RULE.

Divide the numerator by the denominator, for the integral part, and place the remainder, if any, over the denominator, and it will be the mixed quantity required.

EXAMPLES.

1. Let $\frac{17}{5}$, and $\frac{ax+a^2}{x}$ be reduced to whole or mixed quantities.

First, $\frac{17}{5}=17\div5=3\frac{2}{5}$, *the Answer required.*

And $\frac{ax+a^2}{x}=(ax+a^2)\div x=a+\frac{a^2}{x}$ *Ans.*

2. Let $\frac{ab-a^2}{b}$, and $\frac{ay+2y^2}{a+y}$ be reduced to whole or mixed quantities.

First, $\frac{ab-a^2}{b}=(ab-a^2)\div b=a-\frac{a^2}{b}$ *Ans.*

And, $\frac{ay+2y^2}{a+y}=(ay+2y^2)\div(a+y)=y+\frac{y^2}{a+y}$

3. Let $\frac{35}{8}$, and $\frac{3ab-b^2}{a}$ be reduced to whole or mixed quantities. *Ans.* $4\frac{3}{8}$, and $3b-\frac{b^2}{a}$.

4. Let $\frac{2x^2y}{2x}$, and $\frac{a^2+x^2}{a-x}$ be reduced to whole or mixed quantities.

5. Let $\frac{x^2-y^2}{x+y}$, and $\frac{x^3-y^3}{x-y}$ be reduced to whole or mixed quantities.

6. Let $\frac{10x^2-x+3}{5x}$ be reduced to a whole or mixed quantity.

7. Let $\frac{12x^3+3x^2}{4x^3+x^2-4x-1}$ be reduced to a whole or mixed quantity.

PROBLEM III.

To reduce fractions of different denominators, to others of the same value, which shall have a common denominator.

RULE.

Multiply every numerator, feparately, into all the denominators but its own, for the new numerators, and all the denominators together for the common denominator.*

EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{b}{c}$ to fractions of equal values that fhall have a common denominator.

$$a \times c = ac$$
$$b \times b = b^2$$
$$\overline{b \times c = bc}$$

$\frac{ac}{bc}$ and $\frac{b^2}{bc}$ = *fractions required.*

2. Reduce $\frac{a}{b}, \frac{b}{c}$, and $\frac{c}{d}$ to equivalent fractions, having a common denominator.

$$a \times c \times d = acd$$
$$b \times b \times d = b^2d$$
$$c \times b \times c = c^2b$$
$$\overline{b \times c \times d = bcd}$$

$\frac{acd}{bcd}$, $\frac{b^2d}{bcd}$, and $\frac{c^2b}{bcd}$, *fractions required.*

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* When the denominators have a common divifor, it will be better, inftead of multiplying by the whole denominators, to multiply only by thofe parts which arife from dividing by the common divifor.

3. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to equivalent fractions, having a common denominator. *Anſ.* $\frac{2cx}{ac}$ *and* $\frac{ab}{ac}$

4. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to fractions, having a common denominator. *Anſ.* $\frac{ac}{bc}$ *and* $\frac{ab+bb}{bc}$

5. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d to fractions having a common denominator. *Anſ.* $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, and $\frac{6acd}{6ac}$

6. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a+\frac{2x}{a}$, to fractions, having a common denominator.

Anſ. $\frac{9a}{12a}$, $\frac{8ax}{12a}$, and $\frac{12a^2+24x}{12a}$.

7. Reduce $\frac{1}{2}$, $\frac{a^2}{3}$ and $\frac{a^2+x^2}{a+x}$, to fractions, having a common denominator.

8. Reduce $\frac{b}{2a^2}$, $\frac{c}{2a}$, and $\frac{d}{a}$, to equivalent fractions, having a common denominator.

PROBLEM IV.

To find the greateſt common meaſure of a fraction.

RULE.*

1. Range the quantities according to the dimenſions of ſome letters, as is ſhewn in diviſion.

* The ſimple diviſors, in this rule, may be eaſily found, by inſpection.

2. Divide the greater term by the lefs, and the laft divifor by the laft remainder, and fo on till nothing remains; and the divifor laft ufed will be the common meafure required.

Note. All the letters or figures which are common to each term of the divifors, muft be thrown out of them before they are ufed in the operation.

EXAMPLES.

1. To find the greateft common meafure of $\frac{cx+x^2}{ca^2+a^2x}$.

$$cx+x^2)ca^2+a^2x$$
$$\text{or } c+x)ca^2+a^2x(a^2$$
$$ca^2+a^2x$$
$$*$$

Therefore the greateft common meafure is $c+x$.

2. To find the greateft common meafure of $\frac{x^3-b^2x}{x^2+2bx+b^2}$

$$x^2+2bx+b^2)x^3-b^2x(x$$
$$x^3+2bx^2+b^2x$$
$$-2bx^2-2b^2x)x^2+2bx+b^2$$
$$\text{or } x+b)x^2+2bx+b^2(x+b$$
$$x^2+bx$$
$$bx+b^2$$
$$bx+b^2$$
$$*$$

Therefore $x+b$ is the greatest common divisor.

3. To find the greatest common divisor of $\frac{x^2-1}{xy+y}$. *Ans.* $x+1$.

4. To find the greatest common divisor of $\frac{x^4-b^4}{x^5+b^2x^3}$. *Ans.* x^2+b^2.

5. To find the greatest common measure of $\frac{5a^5+10a^4b+5a^3b^2}{a^3b+2a^2b^2+2ab^3+b^4}$.

PROBLEM V.

To reduce a fraction to its lowest terms.

RULE.

1. Find the greatest common measure, as in the last problem.

2. Divide both the terms of the fraction by the common measure thus found, and it will reduce it to its lowest terms as was required.

EXAMPLES.

1. Reduce $\frac{cx+x^2}{ca^2+a^2x}$ to its lowest terms.

$$cx+x^2)ca^2+a^2x$$
$$\text{or } c+x)ca^2+a^2x(a^2$$
$$ca^2+a^2x$$

Here $cx+x^2$ is divided by x which is common to each term.

Therefore $c+x$ *is the greatest common measure.*

and $c+x)\dfrac{cx+x^2}{ca^2+a^2x}=\dfrac{x}{a^2}=$ *fraction required.*

3. Having $\dfrac{x^3-b^2x}{x^2+2bx+b^2}$ given, it is required to reduce it to its least terms.

$$\begin{array}{l}
x^2+2bx+b^2)x^3-b^2x(x \\
\qquad\qquad\quad x^3+2bx^2+b^2x \\
\hline
\qquad\qquad\quad -2bx^2-2b^2x)x^2+2bx+b^2 \\
\qquad\qquad\qquad\quad or\ x+b)x^2+2bx+b^2(x+b \\
\qquad\qquad\qquad\qquad\qquad x^2+bx \\
\hline
\qquad\qquad\qquad\qquad\qquad bx+b^2 \\
\qquad\qquad\qquad\qquad\qquad bx+b^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\quad *
\end{array}$$

Therefore $x+b$ *is the greatest common measure,*

and $x+b)\dfrac{x^3-b^2x}{x^2+2bx+b^2}=\dfrac{x^2-bx}{x+b}=$ *fraction required.*

3. Reduce $\dfrac{x^4-b^4}{x^5-b^2x^3}$ to its lowest terms. *Ans.* $\dfrac{x^2+b^2}{x^3}$.

3. Reduce $\dfrac{x^2-y^2}{x^4-y^4}$ to its lowest terms. *Ans.* $\dfrac{1}{x^2+y^2}$.

. Reduce $\dfrac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its lowest terms.

Reduce $\dfrac{5a^5+10a^4x+5a^3x^2}{a^3x+3a^2x^2+3ax^3+x^4}$ to its lowest terms.

PROBLEM VI.

To add fractional quantities together.

RULE.

1. Reduce the fractions to a common denominator, as in problem the third.

2. Add all the numerators together, and under their sum write the common denominator, and it will give the sum of the fractions required.

EXAMPLES.

1. Having $\frac{x}{2}$ and $\frac{x}{3}$ given, to find their sum.

$$x \times 3 = 3x$$
$$x \times 2 = 2x$$

$$2 \times 3 = 6$$

$$\frac{3x}{6} + \frac{2x}{6} = \frac{5x}{6} = \textit{sum required.}$$

2. Having $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ given, to find their sum.

$$a \times d \times f = adf$$
$$c \times b \times f = cbf$$
$$e \times b \times d = ebd$$

$$b \times d \times f = bdf$$

$$\frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{ebd}{bdf} = \frac{adf + cbf + ebd}{bdf} \textit{ sum required.}$$

*3. Let $a-\frac{3x^2}{b}$ and $b+\frac{2ax}{c}$ be added together.

$\left.\begin{array}{l}3x^2\times c=3cx^2\\2ax\times b=2abx\end{array}\right\}=$ *numerators.*

And $b\times c=bc=$ *common denominator.*

Therefore $a-\frac{3x^2}{b}+b+\frac{2ax}{c}=a-\frac{3cx^2}{bc}+b+\frac{2abx}{bc}=$

$a+b+\frac{2abx-3cx^2}{bc}=$ *sum required.*

4. Add $\frac{3x}{2b}$ and $\frac{x}{5}$ together.* *Ans.* $\frac{15x+2bx}{10b}$.

5. Add $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$ together. *Ans.* $x+\frac{x}{12}$.

6. Add $\frac{x-2}{3}$ and $\frac{4x}{7}$ together. *Ans.* $\frac{19x-14}{21}$.

7. Add $x+\frac{x-2}{3}$ to $3x+\frac{2x-3}{4}$ *Ans.* $4x+\frac{10x-17}{12}$.

8. It is required to add $4x$, $\frac{5x^2}{2a}$ and $\frac{x+a}{2x}$ together.

9. It is required to add $\frac{2x}{3}$, $\frac{7x}{4}$ and $\frac{2x+1}{5}$ together.

10. It is required to add $4x$, $\frac{7x}{9}$ and $2+\frac{x}{5}$ together.

11. It is required to add $3x+\frac{2x}{5}$ and $x-\frac{8x}{9}$ together.

---

* In the addition of mixed quantities, it is beſt to bring the fractional parts only to a common denominator, and to affix their ſum to the ſum of the integers, interpoſing the proper ſign.

PROBLEM VII.

To subtract one fractional quantity from another.

RULE.*

1. Reduce the fractions to a common denominator, as in addition.

2. Subtract the numerators from each other, and under their difference write the common denominator, and it will give the difference of the fractions required.

EXAMPLES.

1. To find the difference of $\frac{x}{3}$ and $\frac{2x}{11}$.

$$x \times 11 = 11x$$
$$2x \times 3 = 6x$$

$$3 \times 11 = 33$$

$$\frac{11x}{33} - \frac{6x}{33} = \frac{5x}{33} = \text{difference required.}$$

2. To find the difference of $\frac{x-a}{3b}$ and $\frac{2a-4x}{5c}$.

$$(x-a) \times 5c = 5cx - 5ac$$
$$(2a-4x) \times 3b = 6ab - 12bx$$

$$3b \times 5c = 15bc$$

$$\frac{5cx-5ac}{15bc} - \frac{6ab-12bx}{15bc} = \frac{5cx-5ac-6ab+12bx}{15cb} = \text{difference required.}$$

* The same rule may be observed for mixed quantities in subtraction as in addition.

3. Required the difference of $\frac{12x}{7}$ and $\frac{3x}{5}$. *Ans.* $\frac{39x}{35}$.

4. Required the difference of $5y$ and $\frac{3y}{8}$. *Ans.* $\frac{37y}{8}$.

5. Required the difference of $\frac{3x}{7}$ and $\frac{2x}{9}$. *Ans.* $\frac{13x}{63}$.

6. Required the difference between $\frac{x+a}{b}$ and $\frac{c}{d}$.

Ans. $\frac{dx+ad-bc}{bd}$.

7. Required the difference of $\frac{3x+a}{5b}$ and $\frac{2x+7}{8}$.

Ans. $\frac{24x+8a-10bx-35b}{40b}$

8. Required the difference of $3x+\frac{x}{b}$ and $x-\frac{x-a}{c}$.

Ans. $2x+\frac{cx-bx+ab}{bc}$.

PROBLEM VIII.

To multiply fractional quantities together.

RULE.*

Multiply the numerators together for a new numerator, and the denominators for a new denominator, and it will give the product required.

---

* 1. When the numerator of one fraction, and the denominator of the other, can be divided by some quantity, which is common to each, the quotients may be used instead of them.

2. When

EXAMPLES.

1. Let it be required to find the product of $\frac{x}{6}$ and $\frac{2x}{9}$.

$$\left.\frac{x\times 2x}{6\times 9}\right\}=\frac{2x^2}{54}=\frac{x^2}{27}=\textit{product required.}$$

2. Required the product of $\frac{x}{2}$, $\frac{4x}{5}$, and $\frac{10x}{21}$.

$$\left.\frac{x\times 4x\times 10x}{2\times 5\times 21}\right\}=\frac{40x^3}{210}=\frac{4x^3}{21}=\textit{product required.}$$

3. Required the product of $\frac{x}{a}$ and $\frac{x+a}{a+c}$.

$$\left.\begin{matrix} x\times(x+a) \\ a\times(a+c)\end{matrix}\right\}=\frac{x^2+ax}{a^2+ac}=\textit{product required.}$$

4. Required the product of $\frac{3x}{2}$ and $\frac{3a}{b}$ *Anſ.* $\frac{9ax}{2b}$.

5. Required the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$ *Anſ.* $\frac{3x^3}{5a}$.

6. Find the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{3ac}{2b}$

Anſ. $9ax$.

7. It is required to find the product of $b+\frac{bx}{a}$ and $\frac{a}{x}$.

Anſ. $\frac{ab+bx}{x}$.

---

2. When a fraction is to be multiplied by an integer, the product is found by multiplying the numerator by it; and if the integer be the ſame with the denominator, the numerator may be taken for the product.

3. When a fraction is to be multiplied by any quantity, it is the ſame thing whether the numerator be multiplied by it, or the denominator divided by it.

8. Required the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$.

9. Required the product of $\frac{x+1}{a}$, and $\frac{x-1}{a+b}$.

PROBLEM IX.

To divide one fractional quantity by another.

RULE.*

Multiply the denominator of the divisor by the numerator of the dividend, for a new numerator; and the numerator of the divisor by the denominator of the dividend, for a new denominator.

Or, which is the same thing, invert the divisor, and proceed exactly as in multiplication.

EXAMPLES.

1. Required the quotient of $\frac{x}{3}$ divided by $\frac{2x}{9}$.

$$\frac{x}{3} \times \frac{9}{2x} = \frac{9x}{6x} = \frac{3}{2} = 1\tfrac{1}{2} = \textit{quotient required.}$$

2. Required the quotient of $\frac{2a}{b}$ divided by $\frac{4c}{d}$.

$$\frac{2a}{b} \times \frac{d}{4c} = \frac{2ad}{4bc} = \frac{ad}{2bc} = \textit{quotient required.}$$

---

1. If the fractions to be divided have a common denominator, take the numerator of the dividend for a new numerator, and the numerator of the divisor for the denominator.

2. When

3. Find the quotient of $\frac{x+a}{2x-2b}$ divided by $\frac{x+b}{5x+a}$.

$\frac{x+a}{2x-2b} \times \frac{5x+a}{x+b} = \frac{5x^2+6ax+a^2}{2x^2-2b^2}$ = *quotient required.*

4. Find the quotient of $\frac{2x^2}{a^3+x^3}$ divided by $\frac{x}{x+a}$.

$\frac{2x^2}{a^3+x^3} \times \frac{x+a}{x} = \frac{2x^2 \times (x+a)}{(a^3+x^3) \times x} = \frac{2x}{x^2-ax+a^2}$ = *quotient required.*

5. Let $\frac{7x}{5}$ be divided by $\frac{12}{13}$ *Anʃ.* $\frac{91x}{60}$.

6. Let $\frac{4x^2}{7}$ be divided by $5x$. *Anʃ.* $\frac{4x}{35}$.

7. Let $\frac{x+1}{6}$ be divided by $\frac{2x}{3}$ *Anʃ.* $\frac{x+1}{4x}$.

8. Let $\frac{x}{x-1}$ be divided by $\frac{x}{2}$ *Anʃ.* $\frac{2}{x-1}$.

9. Let $\frac{5x}{3}$ be divided by $\frac{2a}{3b}$. *Anʃ.* $\frac{5bx}{2a}$.

10. Let $\frac{x-b}{8cd}$ be divided by $\frac{3cx}{4d}$ *Anʃ.* $\frac{x-b}{6c^2x}$.

11. Let $\frac{x^4-b^4}{x^2-2bx+b^2}$ be divided by $\frac{x^2+bx}{x-b}$.

Anʃ. $x+\frac{b^2}{x}$.

---

2. When a fraction is to be divided by any quantity, it is the ſame thing whether the numerator be divided by it, or the denominator multiplied by it.

3. When the two numerators, or the two denominators, can be divided by ſome common quantity, that quantity may be thrown out of each, and the quotients uſed inſtead of the fractions firſt propoſed.

INVOLUTION.

Involution is the raising of powers from any proposed root: or the method of finding the square, cube, biquadrate, &c. of any given quantity.

RULE.*

Multiply the quantity into itself as many times as there are units in the index less one, and the last product will be the power required. *Or,*

Multiply the index of the quantity by the index of the power, and the result will be the same as before.

Note. When the sign of the root is +, all the powers of it will be +; and when the sign is —, all the powers will be +, and all the odd powers —.

EXAMPLES.

$$a\text{, root}\left\{\begin{array}{l} a^2 = \textit{square} \\ a^3 = \textit{cube} \\ a^4 = \textit{4th power} \\ a^5 = \textit{5th power} \end{array}\right. \qquad a^2\text{, root}\left\{\begin{array}{l} a^4 = \textit{square} \\ a^6 = \textit{cube} \\ a^8 = \textit{4th power} \\ a^{10} = \textit{5th power} \end{array}\right.$$

&c. &c.

$$-3a\text{, root}\left\{\begin{array}{l} +\ \ 9a^2 = \textit{square} \\ -\ 27a^3 = \textit{cube} \\ +\ 81a^4 = \textit{4th power} \\ -243a^5 = \textit{5th power} \end{array}\right.$$

---

* The *n*th. power of any product is equal to the *n*th. power of each of the factors, multiplied together.

And the *n*th. power of a fraction, is equal to the *n*th. power of the numerator, divided by the *n*th. power of the denominator.

$$-2ax^2,\ \textit{root}\left\{\begin{array}{l} +\ 4a^2x^4 = \textit{square} \\ -\ 8a^3x^6 = \textit{cube} \\ +16a^4x^8 = \textit{4th power} \\ -32a^5x^{10} = \textit{5th power.} \end{array}\right.$$

$$\frac{x}{a},\ \textit{root}\left\{\begin{array}{l} \frac{x^2}{a^2} = \textit{square} \\ \frac{x^3}{a^3} = \textit{cube} \\ \frac{x^4}{a^4} = \textit{biquadrate} \end{array}\right.$$

$$-\frac{2ax^2}{3b},\ \textit{root}\left\{\begin{array}{l} +\frac{4a^2x^4}{9b^2} = \textit{square} \\ -\frac{8a^3x^6}{27b^3} = \textit{cube} \\ +\frac{16a^4x^8}{81b^4} = \textit{4th power} \end{array}\right.$$

$$\begin{array}{l} x+a = \textit{root} \\ x+a \\ \hline x^2+ax \\ \quad +ax+a^2 \\ \hline x^2+2ax+a^2 = \textit{square} \\ x+a \\ \hline x^3+2ax^2+a^2x \\ \quad +ax^2+2a^2x+a^3 \\ \hline x^3+3ax^2+3a^2x+a^3 = \textit{cube} \\ \hline \end{array}$$

or third power of $x+a$.

EXAMPLES FOR PRACTICE.

1. Required the cube or third power of $2a^2$.

Anſ. $8a^6$.

2. Required the 4th power of $2a^2x$. *Anſ.* $16a^8x^4$

3. Required the 3d power of $-8x^2y^3$.

Anſ. $-512x^6y^9$.

4. To find the biquadrate of $-\frac{2a^2x}{3b^2}$. *Anſ.* $\frac{16a^8x^4}{81b^8}$

5. Required the 5th power of $a-x$.

Anſ. $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$.

SIR ISAAC NEWTON'S RULE *for raising a binomial or residual quantity to any power whatever.**

1. *To find the terms without the co-efficients.* The index of the firſt, or leading quantity, begins with that of the given power, and decreaſes continually by 1, in every term to the laſt; and in the following quantity the indices of the terms are 0, 1, 2, 3, 4, &c.

2. *To find the unciæ, or co-efficients.* The firſt is always 1, and the ſecond is the index of the power: and in general, if the co-efficient of any term be multiplied by the index of the leading quantity, and the product be divided by the number of terms to that place, it will give the co-efficient of the term next following.

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* This rule, expreſſed in general terms, is as follows:

$\overline{a+b}^n = a^n + na^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 + n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3$, &c.

$\overline{a-b}^n = a^n - na^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 - n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3$, &c.

The

Note. The whole number of terms will be one more than the index of the given power; and when both terms of the root are +, all the terms of the power will be +; but if the second term be —, all the odd terms will be +, and the even terms —.

EXAMPLES.

1. Let $a+x$ be involved to the fifth power.

The terms without the co-efficients will be.

$$a^5,\ a^4x,\ a^3x^2,\ a^2x^3,\ ax^4,\ x^5,$$

and the co-efficients will be

$$1,\ 5,\ \frac{5\times4}{2},\ \frac{10\times3}{3},\ \frac{10\times2}{4},\ \frac{5\times1}{5};$$

And therefore the fifth power is

$$a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5.$$

2. Let $x-a$ be involved to the 6th power.

The terms without the co-efficients will be

$$x^6,\ x^5a,\ x^4a^2,\ x^3a^3,\ x^2a^4,\ xa^5,\ a^6,$$

and the co-efficients will be

$$1,\ 6,\ \frac{6\times5}{2},\ \frac{15\times4}{3},\ \frac{20\times3}{4},\ \frac{15\times2}{5},\ \frac{6\times1}{6}.$$

or 1, 6, 15, 20, 15, 6, 1.

And therefore the 6th power of $x-a$ is

$$x^6-6x^5a+15x^4a^2-20x^3a^3+15x^2a^4-6xa^5+a^6.$$

3. Required the 4th power of $x-a$.

Anf. $x^4-4x^3a+6x^2a^2-4xa^3+a^4$.

4. Required the 7th power of $x+a$.

Anf. $x^7+7x^6a+21x^5a^2+35x^4a^3+35x^3a^4+21x^2a^5+7xa^6+a^7$.

The fum of the co-efficients, in every power, is equal to the number 2, raifed to that power. Thus $1+2=2$ for the firft power; $1+2+1=4=2^2$ for the fquare; $1+3+3+1=8=2^3$ for the cube, or third power, &c.

EVOLUTION.

Evolution is the reverſe of involution, or the method of finding the ſquare root, cube root, &c. of any given quantity, whether ſimple or compound.

CASE I.

To find the roots of simple quantities.

RULE.*

Extract the root of the co-efficient for the numerical part, and divide the index of the letter, or letters, by the index of the power, and it will give the root required.

EXAMPLES.

1. Required the ſquare root of $9x^2$, and the cube root of $8x^3$.

Anſ. $\sqrt{9x^2}=3x^{\frac{2}{2}}=3x$; *and* $\sqrt[3]{8x^3}=2x^{\frac{3}{3}}=2x$.

2. Required the ſquare root of $\frac{3x^2y^2}{4a^2}$, and the cube root of $\frac{16x^4y^3}{27a^3}$.

Anſ. $\sqrt{\frac{3x^2y^2}{4a^2}}=\frac{xy}{2a}\sqrt{3}$; *and* $\sqrt[3]{\frac{16x^4y^3}{27a^3}}=\frac{2xy}{3}\sqrt[3]{2x}$.

---

* Any even root of an affirmative quantity may be either $+$ or $-$; thus the ſquare root of $+a^2$ is either $+a$, or $-a$: for $(+a)\times(+a)=+a^2$, and $(-a)\times(-a)=+a^2$.

And an odd root of any quantity will have the ſame ſign as the quantity itſelf; thus the cube root of $+a^3$ is $+a$, and the cube

3. Required the ſquare root of $3a^2x^6$. *Anſ.* $ax^3\sqrt{3}$.
4. Required the cube root of $-125a^3x^6$. *Anſ.* $-5ax^2$.
5. Required the ſquare root of $\frac{9x^2y^2}{4a^2}$ *Anſ.* $\frac{3xy}{2a}$.
6. Required the 4th root of $256a^4x^8$. *Anſ.* $4ax^2$.
7. It is required to find the 5th root of $-32x^5y^{10}$. *Anſ.* $-2xy^2$.

CASE II.

To find the square root of a compound quantity.

RULE.

1. Range the quantities according to the dimenſions of ſome letter, and ſet the root of the firſt term in the quotient.

2. Subtract the ſquare of the root, thus found, from the firſt term, and bring down the two next terms to the remainder for a dividend.

3. Divide the dividend by double the root, and ſet the reſult both in the quotient and diviſor.

4. Multiply the diviſor, thus increaſed, by the term laſt put in the quotient, and ſubtract the product from the dividend; and ſo on, as in common arithmetic.

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cube root of $-a^3$ is $-a$; for $(+a)\times(+a)\times(+a)=+a^3$, and $(-a)\times(-a)\times(-a)=-a^3$.

Any even root of a negative quantity is impoſſible: for neither $(+a)\times(+a)$ nor $(-a)\times(-a)$ can produce $-$.

The *n*th. root of a product is equal to the *n*th. root of each of the factors multiplied together.

And the *n*th. root of a fraction is equal to the *n*th. root of the numerator, divided by the *n*th. root of the denominator.

EXAMPLES.

1. Extract the square root of $x^4-4x^3+6x^2-4x+1$.

$$x^4-4x^3+6x^2-4x+1\left(x^2-2x+1 = root.\right.$$
$$x^4$$

$$2x^2-2x)-4x^3+6x^2$$
$$-4x^3+4x^2$$

$$2x^2-4x+1)2x^2-4x+1$$
$$2x^2-4a+1$$

$$*$$

2. Extract the square root of $4a^4+12a^3x+13a^2x^2+6ax^3+x^4$.

$$4a^4+12a^3x+13a^2x^2+6ax^3+x^4\left(2a^2+3ax+x^2\right.$$
$$4a^4$$

$$4a^2+3ax)12a^3x+13a^2x^2$$
$$12a^3x+9a^2a^2$$

$$4z^2+6ax+x^2)\ 4a^2x^2+6ax^3+x^4$$
$$4a^2x^2+6ax^3+x^4$$

$$*$$

3. Required the square root of $a^4+4a^3x+6a^2x^2+4ax^3+x^4$.

Ans. $a^2+2ax+x^2$.

4. Required the square root of $x^4-2x^3+\frac{3}{2}x^2-\frac{x}{2}+\frac{1}{16}$

Ans. $x^2-x+\frac{1}{4}$.

5. It is required to find the square root of a^2+x^2.

Ans. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-$, &c.

CASE III.

To find the roots of powers in general.

RULE.*

1. Find the root of the first term, and place it in the quotient.

2. Subtract its power from that term, and bring down the second term for a dividend.

3. Involve the root, last found, to the next lowest power, and multiply it by the index of the given power for a divisor.

4. Divide the dividend by the divisor, and the quotient will be the next term of the root.

5. Involve the whole root, and subtract and divide as before; and so on till the whole is finished.

EXAMPLES.

1. Required the square root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$.

$$a^4-2a^3x+3a^2x^2-2ax^3+x^4\,(a^2-ax+x^2$$
$$a^4$$

$$2a^2)-2a^2x$$

$$a^4-2a^3x+\ a^2x^2$$

$$2a^2)2a^2x^2$$

$$a^4-2a^3x+3a^2x^2-2ax^3+x^4$$

---

* As this method, in high powers, is generally thought too laborious, it may not be improper to observe that the roots of compound quantities may sometimes be easily discovered thus:

1. Extract

2. Extract the cube root of $x^6+6x^5-40x^3+96x-64$.

$$x^6+6x^5-40x^3+96x-64(x^2+2x-4$$
$$x^6$$

$$3x^4)6x^5$$

$$x^6+6x^5+12x^4+8x^3$$

$$3x^4)-12x^4$$

$$x^6+6x^5-40x^3+96x-64$$

3. Required the ſquare root of $a^2+2ab+2ac+b^2+2bc+c^2$. *Anſ.* $a+b+c$.

4. Required the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$. *Anſ.* x^2-2x+1.

5. Required the biquadrate root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$. *Anſ.* $2a-3x$.

6. Required the fifth root of $32x^5-80x^4+80x^3-40x^2+10x-1$. *Anſ.* $2x-1$.

---

1. Extract the roots of ſome of the moſt ſimple terms, and connect them together by the ſign + or −, as may be judged moſt ſuitable for the purpoſe.

2. Involve the compound root, thus found, to the proper power, and, if it be the ſame with the given quantity, it is the root required.

3. But if it be found to differ only in ſome of the ſigns, change them from + to −, or from − to +, till its power agrees with the given one throughout.

Thus, in the fifth example, the root $2a-3x$, is the difference of the roots of the firſt and laſt terms; and in the 3d example, the root $a+b+c$ is the ſum of the roots of the 1ſt, 4th, and 6th terms. The ſame may alſo be obſerved of the 6th example, where the root is found from the firſt and laſt terms.

SURDS.

Surds are such quantities as have no exact root, being usually expressed by fractional indices, or by means of the radical sign $\sqrt{}$ placed before them.

Thus, $2^{\frac{1}{2}}$, or $\sqrt{2}$, denotes the square root of 2, and $3^{\frac{2}{3}}$ the cube root of the square of 3; where the numerator shews the power to which the quantity is to be raised and the denominator its root.

PROBLEM I.

To reduce a rational quantity to the form of a surd.

RULE.

Raise the quantity to a power equivalent to that denoted by the index of the surd, and over this new quantity place the radical sign, and it will be of the form required.

EXAMPLES.

1. It is required to reduce 3 to the form of the square root.

First $3\times3=3^2=9$; *whence* $\sqrt{9}$ *the answer.*

2. It is required to reduce $2x^2$ to the form of the cube root.

First, $2x^2\times2x^2\times2x^2=(2x^2)^3=8x^6$; *whence* $\sqrt[3]{8x^6}$ *or* $(8x^6)^{\frac{1}{3}}$ *the answer.*

3. Reduce 5 to the form of the cube root.

Ans. $(125)^{\frac{1}{3}}$.

4. Reduce $\frac{1}{2}xy$ to the form of the square root.

Ans. $\sqrt{\frac{1}{4}x^2y^2}$.

5. Reduce 2 to the form of the 5th root. *Ans.* $(32)^{\frac{1}{5}}$.

6. Let $a^{\frac{1}{2}}$ be reduced to the form of the 6th root.

7. Reduce $a+b$ to the form of the ſquare root, and $a-b$ to the form of the cube root.

PROBLEM II.

To reduce quantities of different indices to other equivalent ones, that shall have a common index.

RULE.

1. Divide the indices of the quantities by the given index, and the quotients will be the new indices for thoſe quantities.

2 Over the ſaid quantities with their new indices, place the given index, and they will make the equivalent quantities required.

3. A common index may alſo be found by reducing the indices of the quantities to a common denominator, and involving each of them to the power denoted by its numerator.

EXAMPLES.

1. Reduce $15^{\frac{1}{4}}$ and $9^{\frac{1}{6}}$ to equivalent quantities having the common index $\frac{1}{2}$.

$$\frac{1}{4} \div \frac{1}{2} = \frac{1}{4} \times \frac{2}{1} = \frac{2}{4} = \frac{1}{2} = \text{1ſt index.}$$

$$\frac{1}{6} \div \frac{1}{2} = \frac{1}{6} \times \frac{2}{1} = \frac{2}{6} = \frac{1}{3} = \text{2d index.}$$

Therefore $(15^{\frac{1}{2}})^{\frac{1}{2}}$ *and* $(9^{\frac{1}{3}})^{\frac{1}{2}}$ = *quantities required.*

2. Reduce a^2 and $x^{\frac{1}{4}}$ to the same common index $\frac{1}{3}$.

$$\frac{1}{2}\div\frac{1}{3}=\frac{2}{1}\times\frac{3}{1}=\frac{6}{1}= \text{1st index.}$$

$$\frac{1}{4}\div\frac{1}{3}=\frac{1}{4}\times\frac{3}{1}=\frac{3}{4}= \text{2d index.}$$

Therefore $(a^6)^{\frac{1}{3}}$ *and* $(x^{\frac{3}{4}})^{\frac{1}{3}}$ = *quantities required.*

3. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.

Anf. $(27)^{\frac{1}{6}}$ and $(4)^{\frac{1}{6}}$.

4. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{4}}$ to the common index $\frac{1}{8}$.

Anf. $(a^4)^{\frac{1}{8}}$ *and* $(b^2)^{\frac{1}{8}}$.

5. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to the same radical fign.

Anf. $\sqrt[nm]{a^m}$ *and* $\sqrt[mn]{b^n}$.

6. Let $(a+b)^{\frac{1}{2}}$ and $(a-b)^{\frac{1}{3}}$ be reduced to a common index.

7. Let $(a+b)^{\frac{1}{3}}$ and $(a-b)^{\frac{1}{4}}$ be reduced to a common index.

PROBLEM III.

To reduce surds to their most simple terms.

RULE.*

Find the greatest power contained in the given furd, and fet its root before the remaining quantities, with the proper radical fign between them.

* When the given furd contains no exact power, it is already in its moft fimple terms.

EXAMPLES.

1. It is required to reduce $\sqrt{48}$ to its most simple terms.

$\sqrt{48}=\sqrt{(16\times3)}=\sqrt{16}\times\sqrt{3}=4\times\sqrt{3}=4\sqrt{3}$, *the answer.*

2. It is required to reduce $\sqrt[3]{108}$ to its most simple terms.

$\sqrt[3]{108}=\sqrt[3]{(27\times4)}=\sqrt[3]{27}\times\sqrt[3]{4}=3\times\sqrt[3]{4}=3\sqrt[3]{4}$, *the answer.*

3. Reduce $\sqrt{125}$ to its most simple terms.

Ans. $5\sqrt{5}$.

4. Reduce $\sqrt{\frac{50}{147}}$ to its most simple terms.

Ans. $\frac{5}{21}\sqrt{6}$.

5. Reduce $\sqrt[3]{243}$ to its most simple terms.

Ans. $3\sqrt[3]{9}$.

6. Reduce $\sqrt[3]{\frac{16}{81}}$ to its most simple terms.

Ans. $\frac{2}{9}\sqrt[3]{18}$.

7. Reduce $\sqrt{98a^2x}$ to its most simple terms.

Ans. $7a\sqrt{2x}$.

8. Reduce $\sqrt{(x^3-a^2x^2)}$ to its most simple terms.

9. Reduce $(a^3x+3a^3x^2)^{\frac{1}{3}}$ to its most simple terms.

10. Reduce $(32a^6-96a^5x)^{\frac{1}{5}}$ to its most simple terms.

PROBLEM IV.

To add surd quantities together.

RULE.

1. Reduce ſuch quantities as have unlike indices to other equivalent ones, having a common index.

2. Bring all fractions to a common denominator, and reduce the quantities to their ſimpleſt terms, as in the laſt problem.

3. Then, if the ſurd part be the ſame in them all, annex it to the ſum of the rational parts, with the ſign of multiplication, and it will give the total ſum required.

But if the ſurd part be not the ſame in all the quantities, they can only be added by the ſigns + and —.

EXAMPLES.

1. It is required to add $\sqrt{27}$ and $\sqrt{48}$ together.

Firſt, $\sqrt{27}=\sqrt{(9\times3)}=3\sqrt{3}$; *and* $\sqrt{48}=\sqrt{(16\times3)}=4\sqrt{3}$;

Whence, $3\sqrt{3}+4\sqrt{3}=(3+4)\sqrt{3}=7\sqrt{3}=$ *ſum required.*

2. It is required to add $\sqrt[3]{500}$, and $\sqrt[3]{108}$ together.

Firſt, $\sqrt[3]{500}=\sqrt[3]{(125\times4)}=5\sqrt[3]{4}$, *and* $\sqrt[3]{108}=\sqrt[3]{(27\times4)}=3\sqrt[3]{4}$;

Whence, $5\sqrt[3]{4}+3\sqrt[3]{4}=(5+3)\sqrt[3]{4}=8\sqrt[3]{4}=$ *ſum required.*

3. Required the sum of $\sqrt{72}$ and $\sqrt{128}$.

Ans. $14\sqrt{2}$.

4. Required the sum of $\sqrt{27}$ and $\sqrt{147}$.

Ans. $10\sqrt{3}$.

5. Required the sum of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$.

Ans. $\frac{19}{30}\sqrt{6}$.

6. Required the sum of $\sqrt[3]{40}$ and $\sqrt[3]{135}$.

Ans. $5\sqrt[3]{5}$.

7. Required the sum of $3\sqrt[3]{\frac{1}{4}}$ and $5\sqrt[3]{\frac{1}{32}}$.

8. Required the sum of $2\sqrt{a^2b}$ and $3\sqrt{64bx^4}$.

9. Required the sum of $9\sqrt{243}$ and $10\sqrt{363}$.

10. It is required to find the sum of $a^{\frac{1}{n}}$ and $a^{\frac{1}{m}}$.

11. Required the sum of $\sqrt{27a^4x}$ and $\sqrt{3a^2x}$.

PROBLEM V.

To subtract, or find the difference of, surd quantities.

RULE.

Prepare the quantities as in the last rule, and the difference of the rational parts annexed to the common surd, will give the difference of the surds required.

But if the quantities have no common surd, they can only be subtracted by means of the sign —.

EXAMPLES.

1. It is required to find the difference of $\sqrt{448}$ and $\sqrt{112}$.

First, $\sqrt{448}=\sqrt{(64\times7)}=8\sqrt{7}$; *and* $\sqrt{112}=\sqrt{(16\times7)}=4\sqrt{7}$;

Whence $8\sqrt{7}-4\sqrt{7}=(8-4)\sqrt{7}=4\sqrt{7}$ *difference required.*

2. It is required to find the difference of $192^{\frac{1}{3}}$ and $24^{\frac{1}{3}}$

First, $192^{\frac{1}{3}}=(64\times3)^{\frac{1}{3}}=4.3^{\frac{1}{3}}$; *and* $24^{\frac{1}{3}}=(8\times3)^{\frac{1}{3}}=2.3^{\frac{1}{3}}$.

Whence $4.3^{\frac{1}{3}}-2.3^{\frac{1}{3}}=(4-2).3^{\frac{1}{3}}=2.3^{\frac{1}{3}}=$ *difference required.*

3. Required the difference of $2\sqrt{50}$ and $\sqrt{18}$.

Ans. $7\sqrt{2}$.

4. Required the difference of $320^{\frac{1}{3}}$ and $40^{\frac{1}{3}}$.

Ans. $2\sqrt[3]{5}$.

5. Required the difference of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$.

Ans. $\frac{4}{45}\sqrt{15}$.

6. Required the difference of $\sqrt[3]{\frac{2}{3}}$ and $\sqrt{\frac{9}{32}}$.

Ans. $\frac{1}{12}\sqrt[3]{18}$.

7. Find the difference of $\sqrt{80a^4x}$ and $\sqrt{20a^2x^3}$.

Ans. $(4a^2-2ax)\sqrt{5x}$.

8. Required the difference of $8\sqrt[3]{a^3b}$ and $\sqrt[3]{a^6b}$.

9. It is required to find the difference of $x^{\frac{1}{n}}$ and $x^{\frac{1}{m}}$.

PROBLEM VI.

To multiply surd quantities together.

RULE.

Reduce the ſurds to the ſame index, and the product of the rational quantities annexed to the product of the ſurds will give the whole product required; which may be reduced to its moſt ſimple terms by Problem 3.

EXAMPLES.

1. It is required to find the product of $3\sqrt{8}$ and $2\sqrt{6}$.

Here, $3\times2\times\sqrt{8}\times\sqrt{6}=6\sqrt{(8\times6)}=6\sqrt{48}=6\sqrt{(16\times3)}=6\times4\times\sqrt{3}=24\sqrt{3}=$ *product required.*

2. It is required to find the product of $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$.

$\frac{1}{2}\times\frac{3}{4}\times\sqrt[3]{\frac{2}{3}}\times\sqrt[3]{\frac{5}{6}}=\frac{3}{8}\times\sqrt[3]{\frac{10}{18}}=\frac{3}{8}\times\sqrt[3]{\frac{15}{27}}=\frac{3}{8}\times\frac{1}{3}\times\sqrt[3]{15}=\frac{3}{24}\sqrt[3]{15}=\frac{1}{8}\sqrt[3]{15}=$ *product required.*

3. Required the product of $5\sqrt{8}$ and $3\sqrt{5}$.

Ans. $30\sqrt{10}$.

4. Required the product of $\frac{1}{2}\sqrt[3]{6}$ and $\frac{2}{3}\sqrt[3]{18}$.

Ans. $\sqrt[3]{4}$.

5. Required the product of $\frac{2}{3}\sqrt{\frac{1}{8}}$ and $\frac{3}{4}\sqrt{\frac{7}{10}}$.

Ans. $\frac{1}{40}\sqrt{35}$.

6. Required the product of $\sqrt[3]{18}$ and $5\sqrt[3]{4}$.

Ans. $10\sqrt[3]{9}$.

7. It is required to find the product of $a^{\frac{1}{3}}$ and $a^{\frac{2}{3}}$.

Anf. $(a^3)^{\frac{1}{3}}$ *or* a.

8. Required the product of $(x+y)^{\frac{1}{2}}$ and $(x+y)^{\frac{1}{3}}$.

9. Required the product of $x+\sqrt{y}$ and $x-\sqrt{y}$.

10. Required the product of $(a+\sqrt{b})^{\frac{1}{2}}$, and, $(a-\sqrt{b})^{\frac{1}{2}}$.

11. It is required to find the product of $x^{\frac{1}{n}}$ and $x^{\frac{1}{m}}$.

PROBLEM VII.

To divide one surd quantity by another.

RULE.

Reduce the furds to the fame index, and the quotient of the rational quantities being annexed to the quotient of the furds, will give the whole quotient required; which may be reduced to its moft fimple terms as before.

EXAMPLES.

1. It is required to divide $8\sqrt{108}$ by $2\sqrt{6}$.

$(8\div2)\sqrt{(108\div6)}=4\sqrt{18}=4\sqrt{(9\times2)}=4\times3\sqrt{2}=12\sqrt{2}$ = *quotient required.*

2. It is required to divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

$8\div4=2$, *and* $512^{\frac{1}{3}}\div2^{\frac{1}{3}}=256^{\frac{1}{3}}=4.4^{\frac{1}{3}}$

Therefore $2\times4\times4^{\frac{1}{3}}=8.4^{\frac{1}{3}}=$ *quotient required.*

3. Let $6\sqrt{10}$ be divided by $3\sqrt{5}$. *Anf.* $2\sqrt{2}$.

4. Let $4\sqrt[3]{1000}$ be divided by $2\sqrt[3]{4}$. *Anſ.* $10\sqrt[3]{2}$.

5. Let $\frac{3}{4}\sqrt{\frac{1}{135}}$ be divided by $\frac{2}{3}\sqrt{\frac{1}{5}}$ *Anſ.* $\frac{1}{8}\sqrt{3}$.

6. Let $\frac{5}{7}\sqrt[3]{\frac{2}{3}}$ be divided by $\frac{2}{5}\sqrt[3]{\frac{3}{4}}$. *Ans.* $\frac{25}{21}\sqrt[3]{3}$.

7. Let $\frac{2}{5}\sqrt{a}$, or $\frac{2}{5}a^{\frac{1}{2}}$ be divided by $\frac{3}{4}a^{\frac{1}{3}}$. *Anſ.* $\frac{8}{15}a^{\frac{1}{6}}$.

8. Let the quantity $x^{\frac{1}{n}}$ be divided by the quantity $x^{\frac{1}{m}}$.

9. Let $x^2-xd-b+d\sqrt{b}$ be divided by $x-\sqrt{b}$.

PROBLEM VIII.

To involve surd quantities to any power.

RULE.

Multiply the index of the quantity by the index of the power to which it is to be raiſed, and to the reſult annex the power of the rational parts, and it will give the power required.

EXAMPLES.

1. It is required to find the ſquare of $\frac{2}{3}a^{\frac{1}{3}}$.

Firſt, $(\frac{2}{3})^2=\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$, *and* $(a^{\frac{1}{3}})^2=a^{\frac{1}{3}\times 2}=a^{\frac{2}{3}}=(a^2)^{\frac{1}{3}}$;

Whence $(\frac{2}{3}a^{\frac{1}{3}})^2=\frac{4}{9}(a^2)^{\frac{1}{3}}=\frac{4}{9}\sqrt[3]{a^2}=$ *ſquare required.*

2. It is required to find the cube of $\frac{5}{7}\sqrt{7}$.

First $(\frac{5}{7})^3 = \frac{5}{7} \times \frac{5}{7} \times \frac{5}{7} = \frac{125}{343}$, *and* $(7^{\frac{1}{2}})^3 = 7^{\frac{3}{2}} = (7^3)^{\frac{1}{2}}$;

Whence, $(\frac{5}{7}\sqrt{7})^3 = \frac{125}{343}(7^3)^{\frac{1}{2}} = \frac{125}{343}(343)^{\frac{1}{2}} =$ *cube required.*

3. Required the square of $3\sqrt[3]{3}$. *Ans.* $9\sqrt[3]{9}$.
4. Required the cube of $2^{\frac{1}{2}}$, or $\sqrt{2}$. *Ans.* $2\sqrt{2}$.
5. Required the 4th power of $\frac{1}{6}\sqrt{6}$. *Ans.* $\frac{1}{36}$.
6. It is required to find the *n*th power of $a^{\frac{1}{m}}$.
7. It is required to find the square of $3+\sqrt{5}$.
8. It is required to find the cube of $2x-3\sqrt{y}$.

PROBLEM IX.

To extract the roots of surd quantities.

RULE.*

Divide the index of the given quantity by the index of the root to be extracted, and to the result annex the root of the rational part, and it will give the root required.

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* The square root of a binomial, or residual, surd $A+B$, or $A-B$, may be found thus: take $\sqrt{(A^2-B^2)}=D$; then $\sqrt{(A+B)} = \sqrt{\frac{A+D}{2}} + \sqrt{\frac{A-D}{2}}$, and $\sqrt{(A-B)} = \sqrt{\frac{A+D}{2}} - \sqrt{\frac{A-D}{2}}$. Thus the square root of $8+2\sqrt{7} = 1+\sqrt{7}$, and the square root of $3-\sqrt{8} = \sqrt{2}-1$: but for the cube, or any higher root, no general rule can be given.

EXAMPLES.

1. It is required to find the ſquare root of $9\sqrt[3]{3}$.

Firſt, $\sqrt{9}=3$, *and*, $(3^{\frac{1}{3}})^{\frac{1}{2}}=3^{\frac{1}{3}\div 2}=3^{\frac{1}{6}}$;

Whence, $(9\sqrt[3]{3})^{\frac{1}{2}}=3\cdot 3^{\frac{1}{6}}=$ *ſquare root required.*

2. It is required to find the cube root of $\frac{1}{8}\sqrt{2}$

Firſt, $\sqrt[3]{\frac{1}{8}}=\frac{1}{2}$, *and* $(\sqrt{2})^{\frac{1}{3}}=2^{\frac{1}{2}\div 3}=2^{\frac{1}{6}}$;

Whence, $(\frac{1}{8}\sqrt{2})^{\frac{1}{3}}=\frac{1}{2}2^{\frac{1}{6}}=$ *cube root required.*

3. Required the ſquare root of 10^3. *Anſ.* $10\sqrt{10}$.

4. Required the cube root of $\frac{8}{27}a^3$. *Anſ.* $\frac{2}{3}a$.

5. Required the 4th root of $3x^2$. *Anſ.* $3^{\frac{1}{4}}.x^{\frac{1}{2}}$.

6. It is required to find the *n*th root of $x^{\frac{1}{m}}$.

7. Required the ſquare root of $x^2-4x\sqrt{a}+4a$.

INFINITE SERIES.

An infinite ſeries is formed from a vulgar fraction, having a compound denominator, or by extracting the root of a ſurd quantity; and is ſuch, as, being continued, would run on *ad infinitum*, in the manner of a decimal fraction.

But, by obtaining a few of the firſt terms, the law of the progreſſion will be manifeſt, ſo that the ſeries may be continued, without actually performing the whole operation.

PROBLEM I.

To reduce fractional quantities into infinite series.

RULE.

Divide the numerator by the denominator, as in common division; and the operation continued, as far as may be thought necessary, will give the series required.

EXAMPLES.

1. Let $\frac{ax}{a-x}$ be proposed to be converted into an infinite series.

$$a-x)\,ax\ldots\left(x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3},\ \&c.\right.$$

$$ax-x^2$$

$$x^2$$

$$x^2-\frac{x^3}{a}$$

$$\frac{x^3}{a}$$

$$\frac{x^3}{a}-\frac{x^4}{a^2}$$

$$\frac{x^4}{a^2}$$

$$\frac{x^4}{a^2}-\frac{x^5}{a^3}$$

$$\frac{x^5}{a^3},\ \&c.$$

2. Let $\frac{1}{1+x}$ be converted into to an infinite ſeries.

$$
\begin{array}{l}
1+x)1\ldots\ldots(1-x+x^2-x^3, \text{ \&c.} \\
\quad 1+x \\
\quad \overline{\quad} \\
\quad -x \\
\quad -x-x^2 \\
\quad \overline{\quad\quad} \\
\quad\quad\quad x^2 \\
\quad\quad\quad x^2+x^3 \\
\quad\quad\quad \overline{\quad\quad} \\
\quad\quad\quad\quad -x^3 \\
\quad\quad\quad\quad -x^3-x^4 \\
\quad\quad\quad\quad \overline{\quad\quad} \\
\quad\quad\quad\quad\quad\quad x^4
\end{array}
$$

3. Let $\frac{b}{a+x}$ be converted into an infinite ſeries.

Anſ. $\frac{b}{a}\times(1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+$, &c.)

4. Let $\frac{a}{a-x}$ be converted into an infinite ſeries.

Anſ. $\frac{b}{a}\times(1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}$, &c.)

5. Let $\frac{1+x}{1-x}$ be converted into an infinite ſeries.

Anſ. $1+2x+2x^2+2x^3+2x^4$, &c.

6. Let $\frac{a^2}{(a+x)^2}$ be converted into an infinite ſeries.

Anſ. $1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}$, &c.

7. Let $\frac{1}{2}$, or its equal $\frac{1}{1+1}$, be converted into an infinite ſeries:

PROBLEM II.

To reduce a compound surd into an infinite series.

RULE.*

Extract the root as in common arithmetic, and the operation, continued as far as may be thought necessary, will give the series required.

EXAMPLES.

1. Extract the square root of a^2+x^2 in an infinite series.

$$\begin{array}{rl}
 & a^2+x^2\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}, \text{ \&c.}\right. \\
 & a^2 \\
 & \overline{} \\
2a+\frac{x^2}{2a}) & x^2 \\
 & x^2+\frac{x^4}{4a^2} \\
 & \overline{\phantom{x^2+\frac{x^4}{4a^2}}} \\
2a+\frac{x^2}{a}-\frac{x^4}{8a^3}) & -\frac{x^4}{4a^2} \\
 & -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\
 & \overline{\phantom{-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}}} \\
2a+\frac{x^2}{a}-\frac{x^4}{4a^2}, \text{ \&c.}) & \frac{x^6}{8a^4}-\frac{x^8}{64a^6} \\
 & \frac{x^6}{8a^4}+\frac{x^8}{16a}, \text{ \&c.} \\
 & \overline{\phantom{\frac{x^6}{8a^4}+\frac{x^8}{16a}, \text{ \&c.}}} \\
 & -\frac{5x^8}{64a}, \text{ \&c.}
\end{array}$$

* This rule is chiefly of use in extracting the square root, the operation being too tedious for the higher powers.

2. Let $\sqrt{1+1}$ be converted into an infinite ſeries.

Anſ. $1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$, &c.

3. Let $\sqrt{a^2-x^2}$ be converted into an infinite ſeries.

Anſ. $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}$, &c.

4. Let $\sqrt[3]{1-x^3}$ be converted into an infinite ſeries.

Anſ. $1-\frac{x^3}{3}-\frac{x^6}{9}-\frac{5x^9}{81}$, &c.

5 Let $\sqrt{a^2+b}$ be converted into an infinite ſeries.

PROBLEM III.

To reduce a binomial ſurd into an infinite ſeries; or to extract any root of a binomial.

RULE.*

Subſtitute the particular letters of the binomial with their proper ſigns, in the following general form, and it will give the root required, obſerving that P is the firſt term, Q the ſecond term divided by the firſt, $\frac{m}{n}$ the index of the power or root; and A, B, C, D, &c. the foregoing terms with their proper ſigns.

$$\overline{P+QP}^{\frac{m}{n}}=P^{\frac{m}{n}}(A)+\frac{m}{n}AQ(B)+\frac{m-n}{2n}BQ(C)+\frac{m-2n}{3n}CQ(D)$$
$$+\frac{m-3n}{4n}DQ(E), \text{ \&c.}$$

---

* Any ſurd may be taken from the denominator of a fraction and put in the numerator, and vice verſa, by only changing the ſign of its index.

EXAMPLES.

1. To extract the square root of r^2-x^2, in an infinite series.

Here P $=r^2$, Q $=-\frac{x^2}{r^2}$, *and* $\frac{m}{n}=\frac{1}{2}$; *therefore* $(r^2-x^2)^{\frac{1}{2}}=r+\left(\frac{1}{2}\times A\times-\frac{x^2}{r^2}\right)+\left(-\frac{1}{4}\times B\times-\frac{x^2}{r^2}\right)+\left(-\frac{3}{6}\times C\times-\frac{x^2}{r^2}\right)+\left(-\frac{5}{8}\times D\times-\frac{x^2}{r^2}\right)$, &c. $=r+\left(-\frac{x^2}{2r^2}.A\right)+\left(\frac{x^2}{4r^2}.B\right)+\left(\frac{3x^2}{6r^2}.C\right)+\left(\frac{5x^2}{8r^2}.D\right)$, &c. $=r-\frac{x^2}{2r^2}A+\frac{x^2}{4r^2}B+\frac{3x^2}{6r^2}C+\frac{5x^2}{8r^2}D$, &c. *which by restoring the values of* A, B, C, D, &c. *becomes* $r-\frac{x^2}{2r}-\frac{x^4}{8r^3}-\frac{x^6}{16r^5}-\frac{5x^8}{128r^7}$, &c. $=$ *series required.*

2. To find the value of $\frac{1}{(a+b)^2}$ or its equal $(a+b)^{-2}$ in an infinite series.

Here P $=a$, Q $=\frac{b}{a}$, *and* $\frac{m}{n}=\frac{-2}{1}$; *therefore* $(a+b)^{-2}=a^{-2}+\left(-\frac{2}{1}.A.\frac{b}{a}\right)+\left(-\frac{3}{2}.B.\frac{b}{a}\right)+\left(-\frac{4}{3}.C.\frac{b}{a}\right)+\left(-\frac{5}{4}.D.\frac{b}{a}\right)$, &c. $=\frac{1}{a^2}+\left(-\frac{2b}{a}.A\right)+\left(-\frac{3b}{2a}.B\right)+\left(-\frac{4b}{3a}.C\right)+\left(-\frac{5b}{4a}.D\right)$, &c. $=\frac{1}{a^2}-\frac{2b}{a}A-\frac{3b}{2a}B-\frac{4b}{3a}C-\frac{5b}{4a}D$, &c. *which, by restoring the values of* A, B, C, D, *becomes* $\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b}{a^4}-\frac{4b}{a^5}+\frac{5b}{a^6}$, &c. $=$ *series required.*

3. To find the value of $\frac{r^2}{r+x}$, in an infinite series.

Anf. $r-x+\frac{x^2}{r}-\frac{x^3}{r^2}+\frac{x^4}{r^3}$, &c.

4. To find the value of $\frac{1}{(a^2-x^2)^{\frac{1}{2}}}$, in an infinite series.

Anf. $\frac{1}{a}+\frac{x^2}{2a^3}+\frac{3x^4}{8a^5}+\frac{15x^6}{48a^7}$ &c.

5. To find the value of $\frac{a^2}{(a+x)^2}$ in an infinite series.

Anf. $1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+\frac{5x^4}{a^4}$, &c.

6. To find the value of $(a^2+b)^{\frac{1}{2}}$ in an infinite series.

Anf. $a+\frac{b}{2a}-\frac{b^2}{8a^3}+\frac{b^3}{16a^5}-\frac{5b^4}{128a^7}$, &c.

7. Find the value of $(a^2-x^2)^{\frac{1}{5}}$ in an infinite series.

Anf. $a^{\frac{2}{5}}\times:1-\frac{x^2}{5a^2}-\frac{2x^4}{25a^4}-\frac{6x^6}{125a^6}$, &c.

8. To find the value of $(a^3-b)^{\frac{1}{3}}$ in an infinite series.

Anf. $a-\frac{b}{3a^2}-\frac{b^2}{9a^5}-\frac{5b^3}{81a^8}-\frac{10b^4}{243a^{11}}$, &c.

9. Required the square root of $\frac{a^2-x^2}{a^2-x^2}$ in an infinite series.

Anf. $1+\frac{x^2}{a^2}-\frac{x^4}{2a^4}+\frac{x^6}{2a^6}$, &c.

10. Required the cube root of $\frac{a^2}{(a^2+x^2)^2}$ in an infinite series.

Anf $\frac{1}{a^{\frac{1}{3}}}\times:1-\frac{2x^2}{3a^2}+\frac{5x^4}{9a^4}-\frac{40x^6}{81a^6}$, &c.

11. Required the value of $\frac{ax}{x^2-ax+x^2}$ in an infinite ſeries.

Anſ. $\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^4}{a^4}-\frac{x^5}{a^5}$, &c.

ARITHMETICAL PROPORTION.

Arithmetical Proportion is the relation which two quantities, of the ſame kind, bear to each other, with reſpect to their difference.

Four quantities are ſaid to be in *arithmetical proportion*, when the difference between the firſt and ſecond is equal to the difference between the third and fourth.

Thus, 3, 7, 12, 16, *and* a, $a+b$, c, $c+b$ *are arithmetically proportional.*

Arithmetical progreſſion is when a ſeries of quantities either increaſe or decreaſe by the ſame common difference.

Thus, 1, 3, 5, 7, 9, 11, &c. *and* a, $a+b$, $a+2b$, $a+3b$, $a+4b$, $a+5b$, &c. *are ſeries in arithmetical progreſſion, whoſe common differences are* 2 *and* b.

The moſt uſeful part of arithmetical proportion is contained in the following theorems:

I. If four quantities be in arithmetical proportion, the ſum of the two means will be equal to the ſum of the two extremes.

Thus if 2, 5, 7, 10, *and* a, b, c, d, *are in arithmetical proportion, then will* $2+10=5+7$, *and* $a+d=b+c$.

II. In any continued arithmetical progreſſion, the ſum of the two extremes, and that of any two terms which are equally diſtant from them, are equal to each other.

Thus, in the series, 2, 4, 6, 8, 10, 12, *&c.* $2+12=4+10=6+8$.

III. The last term of any arithmetical series, is equal to the sum, or difference, of the first term, and the product of the common difference by the number of terms less one; according as the series is increasing or decreasing.

Thus, the 20*th term of* 2, 4, 6, 8, 10, 12, *&c. is* $=2+2(20-1)=2+2\times19=2+31=40$.

And the nth term of a, $a-x$, $a-2x$, $a-3x$, $a-4x$, *&c. is* $=a-(n-1)\times x=a-(n-1)x$.

IV. The sum of any series of quantities in arithmetical progression, is equal to the sum of the two extremes multiplied by half the number of terms.

Thus, the sum of 1, 2, 3, 4, 5, 6, *&c. continued to the* 20*th term, is* $=\frac{(1+20)\times20}{2}=\frac{21\times20}{2}=21\times10=210$,

And the sum of n terms of a, $a+x$, $a+2x$, $a+3x$, *to* $a+mx$, *is* $=(a+a+mx)\frac{n}{2}=(a+\frac{1}{2}mx).n=(a+\frac{n-1}{2}x)n$.

EXAMPLES.

1. The first term of an increasing arithmetical series is 3, the common difference 2, and the number of terms 20; required the sum of the series.

First, $3+2\times(20-1)=3+2\times19=3+38=41=$ *last term.*

And, $(3+41)\times\frac{20}{2}=44\times\frac{20}{2}=44\times10=440=$ *sum required.*

2. The first term of a decreasing arithmetical series is 100, the common difference 3, and the number of terms 34; required the sum of the series.

First, $100-3.(34-1)=100-3.(33)=100-99=1=$ *last term*.

And, $(100+1)\times\frac{34}{2}=101\times\frac{34}{2}=101\times17=1717=$ *sum required*.

3. Required the ſum of the natural numbers 1, 2, 3, 4, 5, 6, &c. continued to 1000 terms.

Anſ. 500500.

4. * Required the ſum of the odd numbers 1, 3, 5, 7, 9, &c. continued to 101 terms. *Anſ.* 10201.

5. How many ſtrokes do the clocks of Venice, which go on to 24 o'clock, ſtrike in the compaſs of a day?

Anſ. 300.

6. The firſt term of a decreaſing arithmetical ſeries is 10, the common difference $\frac{1}{3}$, and the number of terms 21; required the ſum of the ſeries. *Anſ.* 140.

7. One hundred ſtones being placed on the ground, in a ſtraight line, at the diſtance of a yard from each other, how far will a perſon travel who ſhall bring them one by one to a baſket, which is placed one yard from the firſt ſtone. *Anſ.* 5 *miles and* 1300 *yards*.

---

* The ſum of any number of terms (n) of the arithmetical ſeries of odd numbers 1, 3, 5, 7, 9, &c. is equal to the ſquare (n^2) of that number.

That is, if 1, 3, 5, 7, 9, &c. be the numbers,

Then will $1, 2^2, 3^2, 4^2, 5^2$, &c. be the ſums of 1, 2, 3, &c. terms.

For, $0+1$, or the ſum of 1 term $=1^2$, or 1
$1+3$, or the ſum of 2 terms $=2^2$, or 4
$4+5$, or the ſum of 3 terms $=3^2$, or 9
$9+7$, or the ſum of 4 terms $=4^2$, or 16, &c.

Whence it is plain, that, let n be any number whatever, the ſum of n terms will be n^2.

GEOMETRICAL PROPORTION.

Geometrical proportion is that relation of two quantities of the same kind, which arises from considering what part the one is of the other, or how often it is contained in it.

When four quantities are compared together, the first and third are called the *antecedents*, and the second and fourth the *consequents*.

Ratio is the quotient which arises from dividing the antecedent by the consequent, or the consequent by the antecedent.

Four quantities are said to be proportional, when the first is the same part or multiple of the second, as the third is of the fourth.

Thus 2, 8, 3, 12, *and* a, ar, b, br, *are geometrical proportionals.*

Direct proportion is when the same relation subsists between the first term and the second, as between the third and the fourth.

Thus, 3, 6, 5, 10, *and* x, ax, y, ay, *are in direct proportion.*

Reciprocal, or inverse proportion, is when one quantity increases in the same proportion as another diminishes.

Thus, 2, 6, 9, 3, *and* a, ar, br, b, *are in inverse proportion.*

A series of quantities are said to be in geometrical progression, when the first has the same ratio to the second as the second to the third, the third to the fourth, &c.

Thus, 2, 4, 8, 16, 32, 64, *&c. and* a, ar, ar^2, ar^3, ar^4, ar^5, *&c. are series in geometrical progression.*

The most useful part of geometrical proportion, is contained in the following theorems.

I. If four quantities be in geometrical proportion, the product of the two means will be equal to that of the two extremes.

Thus, if 2, 4, 6, 12, *and* a, ar, b, br; *be geometrically proportional, then will* $2\times 12=4\times 6$, *and* $a\times br=b\times ar$.

II. If four quantities be in geometrical proportion, the rectangle of the means divided by either of the extremes will give the other extreme.

Thus, if, 3, 9, 5, 15, *and* x, ax, y, ay, *are geometrically proportional, then will* $\frac{9\times 5}{3}=15$, *and* $\frac{ax\times y}{ay}=x$.

III. In any continued geometrical progression, the product of the two extremes, and that of any other two terms, equally distant from them, will be equal to each other.

Thus, in the series 1, 3, 9, 27, 81, 243, *&c.* $1\times 243=3\times 81=9\times 27$.

IV. In any continued geometrical series, the last term is equal to the first multiplied by such a power of the ratio as is denoted by the number of terms less one.

Thus, in the series 2, 6, 18, 54, 162, *&c.* $2\times 3^4=162$.

V. The sum of any series in geometrical progression is found by multiplying the last term by the ratio, and dividing the difference of this product and the first term by the ratio less one.

Thus, the sum of 2, 4, 8, 16, 32, 64, 128, 256, 512, *is* $\frac{512\times 2-2}{2-1}=1024-2=1022$.

And the sum of n *terms of* a, ar, ar^2, ar^3, ar^4, *&c. to* ar^{n-1}, is $\frac{ar^{n-1}\times r-a}{r-1}=\frac{ar^n-a}{r-1}=\frac{r^n-1}{r-1}a$.

VI. If four quantities, a, b, c, d, or, 2, 6, 5, 15, be proportional, then will any of the following forms of those quantities be also proportional.

1. *directly* $a:b::c:d$ or $2:6::5:15$.
2. *inversely* $b:a::d:c$ or $6:2::15:5$.
3. *alternately* $a:c::b:d$ or $2:5::6:15$.
4. *compoundedly* $a:a+b::c:c+d$ or $2:8::5:20$.
5. *dividedly* $a:b-a::c:d-c$ or $2:4::5:10$.
6. *mixed* $b+a:b-a::d+c:d-c$ or $8:4::20:10$.
7. *by multiplication* $ra:rb::c:d$ or $2.3:6.3::5:15$.
8. *by division* $a\div r:b\div r::c:d$ or $1:3::5:15$.
9. *The numbers* a, b, c, d, *are in harmonical proportion, when* $a:d::a \backsim b:c \backsim d$.

EXAMPLES.

1. The first term of a geometrical series is 1, the ratio 2, and the number of terms 10; what is the sum of the series?

First, $1\times2^9=1\times512=$ *last term*.

And, $\frac{512\times2-1}{2-1}=\frac{1024-1}{1}=1023=$ *sum required*.

2. The first term of a geometric series is $\frac{1}{2}$, the ratio $\frac{1}{3}$, and the number of terms 5; required the sum of the series.

First, $\frac{1}{2}\times\left(\frac{1}{3}\right)^4=\frac{1}{2}\times\frac{1}{81}=\frac{1}{162}=$ *last term*.

And, $\left(\frac{1}{2}-\frac{1}{162}\times\frac{1}{3}\right)\div\left(1-\frac{1}{3}\right)=\left(\frac{1}{2}-\frac{1}{486}\right)\div\frac{2}{3}=\frac{121}{243}\times$
$\frac{3}{2}=\frac{121}{81\times2}=\frac{121}{162}=$ *sum required*.

3. Required the sum of 1, 3, 9, 27, 81, &c. continued to 12 terms. *Ans.* 265720.

4. Required the ſum of $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$, &c. continued to 12 terms.

5. Required the ſum of 1, 2, 4, 8, 16, 32, &c. continued to 100 terms.

SIMPLE EQUATIONS.

An Equation is, when two equal quantities differently expreſſed, are compared together by means of the ſign $=$ placed between them.

Thus $12-5=7$ is an equation, expreſſing the equality of the quantities $12-5$ and 7.

A ſimple equation is that which contains only one unknown quantity without including its power.

Thus, $x-a+b=c$ is a ſimple equation, containing only the unknown quantity x.

Reduction of equations, is the method of finding the value of the unknown quantity; which is ſhewn in the following rules.

RULE I.

Any quantity may be tranſpoſed from one ſide of the equation to the other by changing its ſign.

Thus, if $x+3=7$, *then will* $x=7-3=4$.

And, if $x-4+6=8$, *then will* $x=8+4-6=6$.

Also, if $x-a+b=c-d$, *then will* $x=c-d+a-b$.

And, in like manner, if, $4x-8=3x+20$, *then will* $4x-3x=20+8$, *or* $x=28$.

RULE II.

If the unknown term be multiplied by any quantity, it may be taken away by dividing all the other terms of the equation by it.

Thus, if $ax=ab-a$, *then will* $x=b-1$.

And, if $2x+4=16$, *then will* $x+2=8$, *and* $x=8-2=6$.

In like manner, if $ax+2ba=3c^2$, *then will* $x+2b=\frac{3c^2}{a}$, *and* $x=\frac{3c^2}{a}-2b$.

RULE III.

If the unknown term be divided by any quantity, it may be taken away by multiplying all the other terms of the equation by it.

Thus, if $\frac{x}{2}=5+3$, *then will* $x=10+6=16$.

And, if $\frac{x}{a}=b+c-d$, *then will* $x=ab+ac-ad$.

In like manner, if $\frac{2x}{3}-2=6+4$, *then will* $2x-6=18+12$, *and* $2x=18+12+6=36$ *or* $x=\frac{36}{2}=18$.

RULE IV.

The unknown quantity in any equation may be made free from surds, by transposing the rest of the terms by Rule 1, and then involving each side to such a power as is denoted by the index of the surd.

Thus, if $\sqrt{x}-2=6$, *then will* $\sqrt{x}=6+2=8$, *and* $x=8^2=64$.

And if $\sqrt{4x+16}=12$, *then will* $4x+16=144$, *or* $4x=144-16=128$; *and if both sides of the equation be divided by* 4, x *will be*$=32$.

In like manner, if $\sqrt[3]{2x+3}+4=8$, *then will* $\sqrt[3]{2x+3}=8-4=4$, *and* $2x+3=4^3=64$, *and* $2x=64-3=61$, *or* $x=\frac{61}{2}=30\frac{1}{2}$.

RULE V.

If that side of the equation which contains the unknown quantity be a complete power, it may be reduced by extracting the root of the said power from both sides of the equation.

Thus, if $x^2+6x+9=25$, *then will* $x+3=\sqrt{25}=5$, *or* $x=5-3=2$.

And, if $3x^2-9=21+3$, *then will* $3x^2=21+3+9=33$, *and* $x^2=\frac{33}{3}=11$, *or* $x=\sqrt{11}$.

In like manner, if $\frac{2x^2}{3}+10=20$, *then will* $2x^2+30=60$, *and* $x^2+15=30$, *or* $x^2=30-15=15$, *or* $x=\sqrt{15}$.

RULE VI.

Any analogy or proportion may be converted into an equation, by making the product of the two mean terms equal to that of the two extremes.

Thus, if $3x:16::5:10$, *then will* $3x\times10=16\times5$, *or* $30x=80$, *or* $x=\frac{80}{30}=\frac{8}{3}=2\frac{2}{3}$.

And, if $\frac{2x}{3}:a::b:c$, *then will* $\frac{2cx}{3}=ab$, *and* $2c=3ab$, *or* $x=\frac{3ab}{2c}$.

In like manner, if $12-x:\frac{x}{2}::4:1$, *then will* $12-x=\frac{4x}{2}$ $=2x$, *and* $2x+x=12$, *or* $x=\frac{12}{3}=4$.

RULE VII.

If the ſame quantity be found on both ſides of the equation with the ſame ſign, it may be taken away from each; and if every term in an equation be multiplied or divided by the ſame quantity, it may be ſtruck out of them all.

Thus, if $4x+a=b+a$, *then will* $4x=b$, *and* $x=\frac{b}{4}$.

And, if $3ax+5ab=8ac$, *then will* $3x+5b=8c$, *and* $x=\frac{8c-5b}{3}$.

In like manner, if $\frac{2x}{3}-\frac{8}{3}=\frac{16}{3}-\frac{8}{3}$, *then will* $2x=16$ *and* $x=8$.

MISCELLANEOUS EXAMPLES.

1\. Given $5x-15=2x+6$ to find the value of x.

Firſt, $5x-2x=6+15$,

Or $3x=6+15=21$;

And therefore $x=\frac{21}{3}=7$.

2\. Given $40-6x-16=120-14x$ to find x.

Firſt, $14x-6x=120-40+16$,

Or $8x=136-40=96$;

And therefore $x=\frac{96}{8}=12$.

3. Let $5ax-3b=2dx+c$ be given, to find x.
First, $5ax-2dx=c+3b$.
Or $(5a-2d)\times x=c+3b$.

And therefore $x=\frac{c+3b}{5a-2d}$.

4. $3x^2-10x=8x+x^2$ be given to find x.
First, $3x-10=8+x$.
Or $3x-x=8+10=18$.

And therefore $2x=18$, *or* $x=\frac{18}{2}=9$.

5. Given $6ax^3-12abx^2=3ax^3+6ax^2$, to find x.
First, dividing the whole by $3ax^2$, *we shall have*
$2x-4b=x+2$.
Or $2x-x=2+4b$.
Whence $x=2+4b$.

6. Let $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=10$, be given to find x.

First, $x-\frac{2x}{3}+\frac{2x}{4}=20$.

Also, $3x-2x+\frac{6x}{4}=60$.

And $12x-8x+6x=240$.
Therefore $10x=240$.

And $x=\frac{240}{10}=24$.

7. Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}$ to find x.

First, $x-3+\frac{2x}{3}=40-x+19$.

Also, $3x-9+2x=120-3x+57$.
Therefore, $3x+2x+3x=120+57+9$.

That is, $8x=186$, *or* $x=\frac{186}{8}=23\frac{1}{4}$.

8. Let $\sqrt{\frac{2x}{3}}+5=7$, be given to find x.

First, $\sqrt{\frac{2x}{3}}=7-5=2$.

Whence $\frac{2x}{3}=2^2=4$.

And $2x=12$, *or* $x=\frac{12}{2}=6$.

9. Let $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}$ be given to find x.

First, $x\sqrt{a^2+x^2}+a^2+x^2=2a^2$

Whence $x\sqrt{a^2+x^2}=a^2-x^2$.

And $x^2\times(a^2+x^2)=(a^2-x^2)^2=a^4-2a^2x^2+x^4$

Or $a^2x^2+x^4=a^4-2a^2x^2+x^4$.

Or $a^2x^2+2a^2x^2=a^4$, *or* $3a^2x^2=a^4$

Consequently $x^2=\frac{a^4}{3a^2}$, *and* $x=\sqrt{\frac{a^4}{3a^2}}=a\sqrt{\frac{1}{3}}$.

EXAMPLES FOR PRACTICE.

1. Given $3y-2+24=31$ to find y. *Ans.* $y=3$.
2. Given $x+18=3x-5$ to find x. *Ans.* $x=11\frac{1}{2}$.
3. Given $6-2x+10=20-3x-2$ to find x. *Ans.* $x=2$.
4. Given $x+\frac{1}{2}x+\frac{1}{3}x=11$ to find x. *Ans.* $x=6$.
5. Given $2x-\frac{1}{2}x+1=5x-2$ to find x. *Ans.* $x=\frac{6}{7}$.
6. Given $3ax+\frac{a}{2}-3=bx-a$ to find x.

Ans. $x=\frac{6-3a}{6a-2b}$.

7. Given $\frac{1}{2}x+\frac{1}{3}x-\frac{1}{4}x=\frac{1}{2}$ to find x. *Anf.* $x=\frac{6}{7}$.

8. Given $\sqrt{12+x}=2+\sqrt{x}$ to find x. *Anf.* $x=4$.

9. Given $x+a=\frac{x^2}{a+x}$ to determine x. *Anf.* $x=-\frac{a}{2}$.

10. Given $\sqrt{a^2+x^2}=(b^4+x^4)^{\frac{1}{4}}$ to find x.

Anf. $x=\sqrt{\frac{b^4-a^4}{2a^2}}$.

11. Given $\sqrt{x}+\sqrt{a+x}=\frac{2a}{\sqrt{a+x}}$ to find x.

Anf. $x=\frac{a}{3}$.

12. Given $\frac{a}{1+x}+\frac{a}{1-x}=b$, to find x.

Anf. $x=\sqrt{\frac{b-2a}{b}}$.

13. Given $a+x=\sqrt{a^2+x\sqrt{b^2+x^2}}$, to find x.

Anf. $x=\frac{b^2}{4a}-a$.

PROBLEM I.

To exterminate two unknown quantities, or to reduce the two simple equations containing them, to a single one.

RULE I.

1. Obferve which of the unknown quantities is the leaft involved, and find its value in each of the equations, by the methods already explained.

2. Let the two values thus found be made equal to each other, and there will arife a new equation with only one unknown quantity in it, whofe value may be found as before.

EXAMPLES.

1. Given $\begin{cases} 2x+3y=23 \\ 5x-2y=10 \end{cases}$ to find x and y.

From the first equation $x=\frac{23-3y}{2}$

And from the second $x=\frac{10+2y}{5}$,

Consequently $\frac{23-3y}{2}=\frac{10+2y}{5}$,

Or $115-15y=20+4y$,

Or $19y=115-20=95$;

That is $y=\frac{95}{19}=5$

And $x=\frac{23-15}{2}=4$.

2. Given $\begin{cases} x+y=a \\ x-y=b \end{cases}$ to find x and y.

From the first equation $x=a-y$.

And from the second $x=b+y$.

Therefore $a-y=b+y$, *or*, $2y=a-b$.

Consequently $y=\frac{a-b}{2}$, *and* $x=a-y$.

Or $x=a-\overline{\frac{a-b}{2}}=\frac{a+b}{2}$.

5. Given $\begin{cases} \frac{1}{2}x+\frac{1}{3}y=7 \\ \frac{1}{3}x+\frac{1}{2}y=8 \end{cases}$ to find x and y.

From the first equation $x=14-\frac{2y}{3}$,

And from the second $x=24-\frac{3y}{2}$,

Therefore $14-\frac{2y}{3}=24-\frac{3y}{2}$

And $42-2y=72-\frac{9y}{2}$,

Or $84-4y=144-9y$;

Whence $5y=144-84=60$,

And $y=\frac{60}{5}=12$,

Or $x=14-\frac{2y}{3}=14-\frac{24}{3}=6$.

4. Given $4x+y=34$, and $4y+x=16$ to find x and y. *Anſ.* $x=8$, *and* $y=2$.

5. Given $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$, and $\frac{3x}{4}+\frac{2y}{5}=\frac{61}{120}$, to find x and y. *Anſ.* $x=\frac{1}{2}$ *and* $y=\frac{1}{3}$.

6. Given $x+y=s$, and $x^2-y^2=d$, to find x and y.
Anſ. $x=\frac{s^2+d}{2s}$, *and* $y=\frac{s^2-d}{2s}$.

7. Given $x-y=d$, and $x:y::n:m$, to find x and y.

RULE II.

1. Conſider which of the unknown quantities you would firſt exterminate, and let its value be found in that equation where it is leaſt involved.

2. Subſtitute the value, thus found, for its equal in the other equation, and there will ariſe a new equation, with only one unknown quantity, whoſe value may be found as before.

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+2y=17 \\ 3x-y=2 \end{array}\right\}$ to find x and y.

From the first equation $x=17-2y$.

And this value substituted for x *in the second, gives*

$(17-2y)\times 3-y=2$,

Or $51-6y-y=2$, *or* $51-7y=2$;

That is $7y=51-2=49$;

Whence $y=\frac{49}{7}=7$, *and* $x=17-2y=17-14=3$.

2. Given $\left\{\begin{array}{l} x+y=13 \\ x-y=3 \end{array}\right\}$ to find x and y.

From the first equation $x=13-y$.

And this value being substituted for x *in the 2d, gives* $13-y-y=3$, *or* $13-2y=3$,

That is $2y=13-3=10$,

Whence $y=\frac{10}{2}=5$, *and* $x=13-y=13-5=8$.

3. Given $\left\{\begin{array}{l} a:b::x:y \\ x^2+y^2=c \end{array}\right\}$ to find x and y.

The first analogy turned into an equation,

is $bx=ay$, *or* $x=\frac{ay}{b}$.

And this value of x *being substituted in the 2d,*

gives $\left(\frac{ay}{b}\right)^2+y^2=c$ *or* $\frac{a^2y^2}{b^2}+y^2=c$,

Or $a^2y^2+b^2y^2=cb^2$, *or* $y^2=\frac{cb^2}{a^2+b^2}$,

Whence $y=\left(\frac{cb^2}{a^2+b^2}\right)^{\frac{1}{2}}$, *and* $x=\left(\frac{ca^2}{a^2+b^2}\right)^{\frac{1}{2}}$.

4. Given $2x+3y=16$, and $3x-2y=11$, to find x and y.

Ans. $x=5$ *and* $y=2$.

5. Given $\frac{x}{7}+7y=99$, *and* $\frac{y}{7}+7x=51$, to find x and y.

Anf. $x=7$ *and* $y=14$.

6. Given $\frac{x}{2}-12=\frac{y}{4}+8$, and $\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27$, to find x and y. *Anf.* $x=60$ *and* $y=40$.

7. Given $a:b::x:y$, and $x^3-y^3=d$, to find x and y.

Anf. $\left(\frac{[illegible]}{a^3-b^3}\right)^{\frac{1}{3}}=x$, *and* $\left(\frac{ab^3}{a^3-b^3}\right)^{\frac{1}{3}}=y$.

RULE III.

1. Let the given equations be multiplied or divided by fuch numbers or quantities as will make the term which contains one of the unknown quantities the fame in both equations.

2. Then by adding or fubtracting the equations, according as is required, there will arife a new equation with only one unknown quantity as before.

EXAMPLES.

1. Given $\left\{\begin{matrix} 3x+5y=40 \\ x+2y=14 \end{matrix}\right\}$ to find x and y.

First, multiply the fecond equation by 3, *and it will give* $3x+6y=42$.

Then from the last equation fubtract the first, and it will give $6y-5y=42-40$, *or* $y=2$, *and therefore* $x=14-2y=14-4=10$.

2. Given $\left\{\begin{array}{l}5x-3y=9\\2x+5y=16\end{array}\right\}$ to find x and y.

Let the first equation be multiplied by 2, *and the* 2*d by* 5, *and we shall have* $10x-6y=18$

$$10x+25y=80$$

And if the former of these be subtracted from the latter, it will give $31y=62$, *or* $y=\frac{62}{31}=2$.

And consequently $x=\frac{9+3y}{5}$, *by the first equation.*

$$Or\ x=\frac{9+6}{5}=\frac{15}{5}=3.$$

Another method.

Multiply the first equation by 5, *and the second by* 3, *and we shall have* $\left\{\begin{array}{l}25x-15y=45\\6x+15y=48\end{array}\right.$

Now, let these two equations be added together, and the sum will be $31x=93$, *or* $x=\frac{93}{31}=3$.

And consequently $y=\frac{16-2x}{5}$ *by the second equation,*

Or $y=\frac{16-6}{5}=\frac{10}{5}=2$ *as before.*

MISCELLANEOUS EXAMPLES.

1. Given $\frac{x+2}{3}+8y=31$, and $\frac{y+5}{4}+10x=192$, to find x and y. *Ans.* $x=19$ *and* $y=3$.

2. Given $\frac{2x-y}{2}+14=18$, and $\frac{2y+x}{3}+16=19$, to find x and y. *Ans.* $x=5$ *and* $y=2$.

3. Given $\frac{2x+3y}{6}+\frac{x}{3}=8$, and $\frac{7y-3x}{2}-y=11$, to find x and y. *Ans.* $x=6$ *and* $y=8$

4. Given $ax+by=c$, and $dx+ey=f$, to find x and y.

Ans. $x=\frac{ce-bf}{ae-bd}$ *and* $y=\frac{af-dc}{ae-bd}$.

PROBLEM II.

To exterminate three unknown quantities, or to reduce the three simple equations, containing them, to a single one.

RULE.

1. Let x, y, and z, be the three unknown quantities to be exterminated.

2. Find the value of x from each of the three given equations.

3. Compare the firſt value of x with the ſecond, and an equation will ariſe involving only y and z.

4. In like manner, compare the firſt value of x with the third, and another equation will ariſe involving only y and z.

5. Find the values of y and z from theſe two equations, according to the former rules, and x, y, and z, will be exterminated as required.

Note. Any number of unknown quantities may be exterminated in nearly the ſame manner; but there are often much ſhorter methods for performing the operation, which will be beſt lea[illegible]om practice.

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y+z=29 \\ x+2y+3z=62 \\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=10 \end{array}\right\}$ to find x, y, and z.

From the first $x=29-y-z$.
From the second $x=62-2y-3z$.
From the third $x=20-\frac{2y}{3}-\frac{z}{2}$,
Whence $29-y-z=62-2y-3z$,
And $29-y-z=20-\frac{2y}{3}-\frac{z}{2}$.
Also from the first of these $y=33-2z$.
And from the second $y=27-\frac{3z}{2}$,
Therefore $33-2z=27-\frac{3z}{2}$, *or* $z=12$,
Whence also $y=33-2z=9$.
And $x=29-7-z=8$.

2. Given $\left\{\begin{array}{l} \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=62 \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=47 \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=38 \end{array}\right\}$ to find x, y and z.

First, the given equations, cleared of fractions, become

$$12x+8y+6z=1488$$
$$20x+15y+12z=2820$$
$$30x+24y+20z=4560$$

And, if the second of these equations be subtracted from double the first, and three times the third from five times the second, we shall have

$$4x+y=156$$
$$10x+3y=420$$

And again, if the second of these be subtracted from three times the first, it will give

$$12x-10x=468-420, \text{ or } x=\frac{48}{2}=24,$$

Therefore $y=156-4x=60$, *and* $z=\frac{1488-8y-12x}{6}$ $=120$.

3. Given $x+y+z=53$, $x+2y+3z=105$, and $x+3y+4z=134$, to find x, y and z.

Ans. $x=24$, $y=6$, and $z=23$.

4. Given $x+y=a$, $x+z=b$, and $y+z=c$, to find x, y and z.

5. Given $\left\{\begin{array}{l} ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p \end{array}\right\}$ to find x, y, and z.

A COLLECTION OF QUESTIONS PRODUCING SIMPLE EQUATIONS.

1. To find two numbers, such that their sum shall be 40, and their difference 16.

Let x *denote the least of the two numbers required,*
Then will $x+16=$ *to the greater,*
And $x+x+16=40$ *by the question.*
That is $2x=40-16=24$,
Or $x=\frac{24}{2}=12=$ *least number.*

And $x+16=12+16=28=$ *greater number required.*

2. What number is that whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 16?

Let $x=$ *number required,*

Then will its $\frac{1}{3}$ *part be* $\frac{x}{3}$, *and its* $\frac{1}{4}$ *part* $\frac{x}{4}$;

And therefore $\frac{x}{3}-\frac{x}{4}=16$ *by the question,*

That is, $x-\frac{3x}{4}=48$, *or* $4x-3x=192$;

Whence $x=192$ *the number required.*

3. Divide 1000l. between A, B, and C, so that A shall have 72l. more than B, and C 100l. more than A.

Let $x=$ B's *share of the given sum,*
Then will $x+72=$ A's *share,*
And $x+172=$ C's *share,*
And the sum of all their shares $x+x+72+x+172$.
Or $3x+244=1000$ *by the question,*
That is $3x=1000-244=756$,
Or $x=\frac{756}{3}=252$l. $=$ B's *share.*
And $x+72=252+72=324$l. $=$ A's *share.*
And $x+172=252+172=424$l. $=$ C's *share.*

252l.
324l.
424l.

1000l. *the proof.*

4. A prize of 1000l. is to be divided between two persons, whose shares therein are in the proportion of 7 to 9; required the share of each.

Let $x+$ *first person's share,*
Then will $1000-x=$ *second person's share,*
And $x:1000-x::7:9$, *by the question,*

That is $9x=(1000-x)\times 7=7000-7x$,

Or $9x+7x=16x=7000$,

Whence $x=\frac{7000}{16}=437$l. 10s. = *1st share*,

And $1000-x=1000-437$l. 10*s.* $=562$l. 10*s.* *2d share.*

5. The paving of a square at 2s. a yard, costs as much as the inclosing it at 5s. a yard; required the side of the square.

Let $x=$ *side of the square sought*,

Then $4x=$ *yards of inclosure*,

And $x^2=$ *yards of pavement;*

Whence $4x\times 5=20x=$ *price of inclosing.*

And $x^2\times 2=2x^2=$ *price of paving*,

And $2x^2=20x$ *by the question.*

Therefore $2x=20$, *and* $x=10=$ *length of the side required.*

6. A labourer engaged to serve for 40 days upon these conditions, that for every day he worked he was to receive 20d. but for every day he played, or was absent, he was to forfeit 8d. now at the end of the time he had to receive 1l. 11s. 8d. it is required to find how many days he worked, and how many he was idle.

Let x *be the number of days he worked*,

Then will $40-x$ *be the number of days he was idle.*

Also $x\times 20=20x=$ *sum earned*,

And $(40-x)\times 8=320-8x=$ *sum forfeited*,

Whence $20x-(320-8x)=380$*d.* (1l. 11s. 8d.) *by the question; that is*, $20-320+8x=380$,

Or $28x=380+320=700$,

And $x=\frac{700}{28}=25=$ *number of days he worked*

And $40-x=40-25=15=$ *number of days he was idle.*

7. Out of a cask of wine, which had leaked away $\frac{1}{3}$ 21 gallons were drawn; and then, being gauged, it appeared to be half full; how much did it hold?

Let it be supposed to have held x gallons,

Then it would have leaked $\frac{x}{3}$ *gallons,*

And consequently there had been taken away $21+\frac{x}{3}$ *gallons.*

But $21+\frac{x}{3}=\frac{x}{2}$ *by the question.*

That is $63+x=\frac{3x}{2}$,

Or $126+2x=3x$,
Hence $3x-2x=126$
Or $x=126=$ *number of gallons required.*

8. What fraction is that to the numerator of which, if 1 be added, the value will be $\frac{1}{3}$; but if 1 be added to the denominator, its value will be $\frac{1}{4}$?

Let the fraction be represented by $\frac{x}{y}$

Then will $\frac{x+1}{y}=\frac{1}{3}$,

And $\frac{x}{y+1}=\frac{1}{4}$,
Or $3x+3=y$,
And $4x=y+1$,
Hence $4x-3x-3=y+1-y$,
That is $x-3=1$,
Or $x=4$, *and* $y=3x+3=12+3=15$.

So that $\frac{4}{15}=$ *fraction required.*

9. A market woman bought in a certain number of eggs at 2 a penny, and as many at 3 a penny, and sold them all out again at the rate of 5 for two-pence, and by so doing lost 4d. what number of eggs had she?

Let $x =$ *number of eggs of each sort,*

Then will $\frac{x}{2} =$ *price of the first sort.*

And $\frac{x}{3} =$ *price of the second sort.*

But $5 : 2 :: 2x$ (*the whole number of eggs*) $: \frac{4x}{5}$,

Whence $\frac{4x}{5}$ *price of both sorts, at five for 2d.*

And $\frac{x}{2} + \frac{x}{3} - \frac{4x}{5} = 4$ *by the question.*

That is $x + \frac{2x}{3} - \frac{8x}{5} = 8$;

Or $3x + 2x - \frac{24x}{5} = 24$;

Or $15x + 10x - 24x = 120$,

Whence $x = 120 =$ *number of eggs of each sort required.*

10. If A can do a piece of work alone in ten days and B in thirteen; set them both about it together, in what time will it be finished?

Let the time sought be denoted by x,

Then 10 *days* : 1 *work* :: x *days* : $\frac{x}{10}$

And 13 *days* : 1 *work* :: x *days* : $\frac{x}{13}$

Hence $\frac{x}{10} =$ *part done by* A *in* x *days;*

And $\frac{x}{13} =$ *part done by* B *in* x *days.*

Consequently $\frac{x}{10}+\frac{x}{13}=1$;

That is $\frac{13x}{10}+x=13$, *or* $13x+10x=130$;

And therefore $23x=130$, *or* $x=\frac{130}{23}=5\frac{15}{23}$ *days, the time required.*

11. If one agent A, alone, can produce an effect e, in the time a, and another agent B, alone, in the time b; in what time will they both together produce the same effect?

Let the time sought be denoted by x.

Then $a:e::x:\frac{ex}{a}=$ *part of the effect produced by* A,

And $b:e::x:\frac{ex}{b}$ *part of the effect produced by* B,

Whence $\frac{ex}{a}+\frac{ex}{b}=e$ *by the question;*

Or $\frac{x}{a}+\frac{x}{b}=1$;

That is $x+\frac{ax}{b}=a$;

Or $bx+ax=ba$;

And consequently $x=\frac{ba}{b+a}=$ *time required.*

QUESTIONS FOR PRACTICE.

1. What two numbers are those whose difference is 7, and sum 33. *Ans.* 13 *and* 20.

2. To divide the number 75 into two such parts, that three times the greater may exceed seven times the less by 15. *Ans.* 54 *and* 21.

3. In a mixture of wine and cyder, $\frac{1}{2}$ of the whole *plus* 25 gallons was wine, and $\frac{1}{3}$ part *minus* 5 gallons was cyder; how many gallons were there of each?

Ans. 85 *of wine, and* 35 *cyder.*

4. A bill of 120l. was paid in guineas and moidores; and the number of pieces of both sorts that were used was just 100; how many were there of each?

Ans. 50 *of each.*

5. Two travellers set out at the same time from London and York, whose distance is 150 miles; one of them goes 8 miles a day, and the other 7; in what time will they meet? *Ans. in* 10 *days.*

6. At a certain election 375 persons voted, and the candidate chosen had a majority of 91; how many voted for each? *Ans.* 233 *for one, and* 142 *for the other.*

7. What number is that from which, if 5 be substracted, $\frac{2}{3}$ of the remainder will be 40? *Ans.* 65.

8. A post is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and 10 feet above the water; what is its whole length?

Ans. 24 *feet.*

9. There is a fish whose tail weighs 9*lb.* his head weighs as much as his tail and half his body, and his body weighs as much as his head and his tail; what is the whole weight of the fish? *Ans.* 72*lb.*

10. After paying away $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purse; what was in it at first?

Ans. 120 *guineas.*

11. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140; what is the age of each? *Ans.* A's=84, B's=42, *and* C's=14.

12. Two persons, A and B, lay out equal sums of money in trade, A gains 126l. and B loses 87l. and A's money is now double of B's; what did each lay out?

Ans. 300l.

13. A perſon bought a chaiſe, horſe, and harneſs, for 60l. the horſe came to twice the price of the harneſs, and the chaiſe to twice the price of the horſe and harneſs; what did he give for each?

Ans. 13*l.* 6*s.* 8*d.* *for the horse*, 6*l.* 13*s.* 4*d.* *for the harness, and* 40*l.* *for the chaise.*

14. Two perſons, A and B, have both the ſame income: A ſaves $\frac{1}{5}$ of his yearly, but B, by ſpending 50l. *per annum* more than A, at the end of 4 years finds himſelf 100l. in debt: what is their income? *Ans.* 125*l.*

15. A perſon has two horſes, and a ſaddle worth 50l. now if the ſaddle be put on the back of the firſt horſe, it will make his value double that of the ſecond; but if it be put on the back of the ſecond, it will make his value triple that of the firſt; what is the value of each horſe? *Ans. One* 30*l. and the other* 40*l.*

16. To divide the number 36 into three ſuch parts that $\frac{1}{2}$ of the firſt, $\frac{1}{3}$ of the ſecond, and $\frac{1}{4}$ of the third, may be all equal to each other?

Ans. The parts are 8, 12, *and* 16.

17. A footman agreed to ſerve his maſter for 8l. a year and a livery, but was turned away at the end of 7 months, and received only 2l. 13. 4d. and his livery; what was its value? *Ans.* 4*l.* 16*s.*

18. A perſon was deſirous of giving 3d. a-piece to ſome beggars, but found that he had not money enough in his pocket by 8d. he therefore gave them each 2d. and had then 3d. remaining; required the number of beggars? *Ans.* 11.

19. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3; but two of the greyhound's leaps are as much as three of the hare's; how many leaps muſt the greyhound take to catch the hare? *Ans.* 300.

20. A person in play lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which he lost $\frac{1}{3}$ of what he then had, and then won 2 shillings; lastly he lost $\frac{1}{7}$ of what he then had: and, this done, found he had but 12s. remaining; what had he at first? *Ans.* 20s.

21. To divide the number 90 into four such parts, that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2; the sum, difference, product, and quotient shall be all equal to each other.

Ans. The parts are 18, 22, 10, *and* 40, *respectively.*

22. The hour and minute hand of a clock are exactly together at 12 o'clock; when are they next together?

Ans. 1 *ho.* $\frac{5}{11}$ *min.*

23. There is an island 73 miles in circumference, and three footmen all start together to travel the same way about it: A goes 5 miles a day, B 8, and C 10; when will they all come together again? *Ans.* 73 *days.*

24. How much foreign brandy at 8s. *per* gallon, and British spirits at 3s. *per* gallon, must be mixed together so that in selling the compound at 9s. *per* gallon, the distiller may clear 30 *per cent?*

Ans. 51 *gallons of brandy, and* 14 *of spirits.*

25. A man and his wife usually drank out a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days; how many days would the man alone be in drinking it? *Ans.* 20 *days.*

26. If A and B together can perform a piece of work in 8 days; A and C together in 9 days; and B and C in 10 days; how many days will it take each person to perform the same work alone?

Ans. A $14\frac{34}{49}$ *days,* B $17\frac{23}{41}$, *and* C $23\frac{7}{31}$.

27. If three agents, A, B, and C, can produce the effects a, b, c, in the times e, f, g, respectively; in what time would they jointly produce the effect d?

Ans. $d \div \left(\frac{a}{e} + \frac{b}{f} + \frac{c}{g}\right)$ *time.*

QUADRATIC EQUATIONS.

A ſimple quadratic equation is that which involves the ſqreua of the unknown quantity only.

An adfected quadratic equation is that which involves the ſquare of the unknown quantity, together with the product that ariſes from multiplying it by ſome known quantity.

Thus $ax^2=b$, is a ſimple quadratic equation,

And $ax^2+bx=c$, is an adfected quadratic equation.

The rule for a ſimple quadratic equation has been given already.

All adfected quadratic equations fall under the three following forms:

1. $x^2+ax=b$
2. $x^2-ax=b$
3. $x^2-ax=-b$.

The rule for finding the value of x, in each of theſe equations, is as follows:

RULE.*

1. Tranſpoſe all the terms which involve the unknown quantity to one ſide of the equation, and the known terms to the other, and let them be ranged according to their dimenſions.

* The ſquare root of any quantity may be either + or —, and therefore all quadratic equations admit of two ſolutions. Thus the ſquare root of $+n^2$ is $+n$ or $-n$; for $(+n)\times(+n)$ or $(-n)\times(-n)$ are each equal to $+n^2$, but the ſquare root of $-n^2$ or $\sqrt{-n^2}$, is imaginary or impoſſible.

2. When the ſquare of the unknown quantity has any co-efficient prefixed to it, let all the reſt of the terms be divided by that co-efficient.

3. Add the ſquare of half the co-efficient of the ſecond term to both ſides of the equation, and that ſide which involves the unknown quantity will then be a complete ſquare.

4. Extract the ſquare root of both ſides of the equation, and the value of the unknown quantity will be determined as was required.

Note, 1. The ſquare root of the firſt ſide of the equation is always equal to the unknown quantity, with half the co-efficient of the ſecond term ſubjoined to it.

2. All equations, in which there are two terms involving the unknown quantity, and which have the index of the one juſt double that of the other, are ſolved like quadratics, by completing the ſquare.

Thus, $x^4+ax^2=b$, *or* $x^{2n}+ax^n=b$, *are the ſame as quadratics, and the value of the unknown quantity may be determined accordingly.*

So, in the firſt form, ($x^2+ax=b$) where $x+\frac{1}{2}a$ is found $=\sqrt{(b+\frac{1}{4}a^2)}$ the root may be either $\sqrt{+}(b+\frac{1}{4}a^2)$ or $-\sqrt{(b+\frac{1}{4}a^2)}$ ſince either of them being multiplied by itſelf will produce $b+\frac{1}{4}a^2$. And this ambiguity is expreſſed by writing the uncertain ſign $\pm$ before $\sqrt{(b+\frac{1}{4}a^2)}$; thus $x=\pm\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}a$.

In this form, where $x=\pm\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}a$ the firſt value of x, viz. $x=+\sqrt{(b-\frac{1}{4}a^2}-\frac{1}{2}a$ is always affirmative; for ſince $\frac{1}{4}a^2+b$ is greater than $\frac{1}{4}a^2$, the greateſt ſquare muſt neceſſarily have the greateſt ſquare root; therefore $\sqrt{(b+\frac{1}{4}a^2)}$ will always be greater than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$; and conſequently $+\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}a$ will always be affirmative.

EXAMPLES.

1. Given $x^2+4x=140$ to find x.

First, $x^2+4x+4=140+4=144$ *by completing the square.*

Then $\sqrt{(x^2+4x+4)}=\sqrt{144}$ *by extracting the root;*

Or, which is the same thing, $x+2=12$,

And therefore $x=12-2=10$.

2. Given $x^2-6x+8=80$, to find x.

First $x^2-6x=80-8=72$ *by transposition.*

Then $x^2-6x+9=72+9=81$ *by completing the square.*

And $x-3=\sqrt{81}=9$ *by extracting the root.*

Therefore $x=9+3=12$.

The ſecond value, viz. $x=-\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}a$, will always be negative, becauſe it is compoſed of two negative terms. Therefore when $x^2+ax=b$, we ſhall have $x=+\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}a$ for the affirmative value of x, and $x=-\sqrt{(b+\frac{1}{4}a^2)}-\frac{1}{2}a$ for the negative value of x.

In the ſecond form, where $x=\pm\sqrt{(b+\frac{1}{4}a^2)}+\frac{1}{2}a$ the firſt value, viz. $x=+\sqrt{(b+\frac{1}{4}a^2)}+\frac{1}{2}a$ is always affirmative, ſince it is compoſed of two affirmative terms. The ſecond value, viz $x=-\sqrt{(b+\frac{1}{4}a^2)}+\frac{1}{2}a$ will always be negative; for ſince $b+\frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2$, $\sqrt{(b+\frac{1}{4}a^2)}$ will be greater than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$; and conſequently $-\sqrt{(b+\frac{1}{4}a^2)}+\frac{1}{2}a$ is always a negative quantity.

Therefore when $x^2-ax=b$ we ſhall have $x=+\sqrt{(b+\frac{1}{4}a^2)}+\frac{1}{2}a$ for the affirmative value of x; and $x=-\sqrt{(b+\frac{1}{4}a^2)}+\frac{1}{2}a$ for the negative value of x; ſo that in both the firſt and ſecond forms, the unknown quantity has always two values, one of which is poſitive, and the other negative.

3. Given $2x^2+8x-20=70$, to find x.
First, $2x^2+8x=70+20=90$ *by transposition.*
Then $x^2+4x=45$ *by dividing by* 2,
And $x^2+4x+4=49$ *by completing the square;*
Whence $x+2=\sqrt{49}=7$ *by extracting the root.*
And consequently $x=7-2=5$.

4. Given $3x^2-3x+6=5\frac{1}{3}$ to find x.
Here $x^2-x+2=1\frac{7}{9}$ *by dividing by* 3,
And $x^2-x=1\frac{7}{9}-2$ *by transposition;*
Also $x^2-x+\frac{1}{4}=1\frac{7}{9}-2+\frac{1}{4}=\frac{1}{36}$ *by completing the square;*
And $x-\frac{1}{2}=\sqrt{\frac{1}{36}}=\frac{1}{6}$ *by evolution.*
Therefore $x=\frac{1}{6}+\frac{1}{2}=\frac{2}{3}$ *the answer.*

5. Given $\frac{x^2}{2}-\frac{x}{3}+20\frac{1}{2}=42\frac{2}{3}$ to find x.

Here $\frac{x^2}{2}-\frac{x}{3}=42\frac{2}{3}-20\frac{1}{2}=22\frac{1}{6}$ *by transposition.*

And $x^2-\frac{2x}{3}=44\frac{1}{3}$ *by multiplying by* 2

Whence $x^2-\frac{2x}{3}+\frac{1}{9}=44\frac{1}{3}+\frac{1}{9}=44\frac{4}{9}$ *by completing the square,*

And $x-\frac{1}{3}=\sqrt{44\frac{4}{9}}=6\frac{2}{3}$ *by evolution.*
Therefore $x=6\frac{2}{3}+\frac{1}{3}=7$ *the answer.*

In the third form, where $x=\pm\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$ both the values of x will be positive, supposing $\frac{1}{4}a^2$ is greater than b. For the first value, viz. $x=+\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$ will then be affirmative, being composed of two affirmative terms.

The second value, viz. $x=-\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$ is affirmative; for since $\frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2-b$, $\sqrt{\frac{1}{4}a^2}$ or $\frac{1}{2}a$ is greater than $\sqrt{(\frac{1}{4}a^2-b)}$; and consequently $-\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$ will always be an affirmative quantity. Therefore, when $x^2-ax=-b$, we shall have $x=+\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$, and also $x=-\sqrt{(\frac{1}{4}a^2-b)}+\frac{1}{2}a$ for the affirmative values of x.

6. Given $ax^2+bx=c$, to find x.

First, $x^2+\frac{b}{a}x=\frac{c}{a}$ *by division;*

Then $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{c}{a}+\frac{b^2}{4a^2}$ *by completing the square;*

And $x+\frac{b}{2a}=\surd\left(\frac{c}{a}+\frac{b^2}{4a^2}\right)=\surd\left(\frac{4ac+b^2}{4a^2}\right)$ *by evolution;*

Therefore $x=\pm\surd\left(\frac{4ac+b^2}{4x^2}\right)-\frac{b}{2a}$.

7. Given $ax^2-bx+c=d$, to find x.

Here, $ax^2-bx=d-c$ *by transposition,*

And $x^2-\frac{b}{a}x=\frac{d-c}{a}$ *by division.*

Also $x^2-\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{d-c}{a}+\frac{b^2}{4a^2}$ *by completing the square;*

And $x-\frac{b}{2a}=\pm\surd\left(\frac{d-c}{a}+\frac{b^2}{4a^2}\right)$ *by evolution.*

Therefore $x=\frac{b}{2a}\pm\surd\left(\frac{d-c}{a}+\frac{b^2}{4a^2}\right)$.

8. Given $x^4+2ax^2=b$, to find x.

Here, $x^4+2ax^2+a^2=b+a^2$ *by completing the square,*

And $x^2+a=\surd(b+a^2)$ *by evolution;*

Whence $x^2=\surd(b+a^2)-a$,

And consequently $x=\surd\overline{-a+\surd(b+a^2)}$.

But in this third form, if b be greater than $\frac{1}{4}a^2$, the solution of the proposed question will be impossible. For since the square of any quantity, whether that quantity be affirmative or negative, is always affirmative, the square root of a negative quantity is impossible, and cannot be assigned. But if b be greater than $\frac{1}{4}a^2$, then $\frac{1}{4}a^2-b$ is a negative quantity; and therefore $\surd(\frac{1}{4}a^2-b)$ is impossible, or imaginary; consequently in that case $x=\frac{1}{2}a\pm\surd(\frac{1}{4}a^2-b)$ is always impossible, or imaginary.

9. Given $ax^n - bx^{\frac{n}{2}} - c = -d$, to find x.

First, $ax^n - bx^{\frac{n}{2}} = c - d$ *by transposition,*

And $x^n - \frac{b}{a}x^{\frac{n}{2}} = \frac{c-d}{a}$ *by division.*

Also $x^n - \frac{b}{a}x^{\frac{n}{2}} + \frac{b^2}{4a^2} = \frac{c-d}{a} + \frac{b^2}{4a^2}$ *by completing the square.*

And $x^{\frac{n}{2}} - \frac{b}{2a} = \sqrt{\left(\frac{c-d}{a} + \frac{b^2}{4a^2}\right)}$ *by evolution.*

Therefore $x^{\frac{n}{2}} = \frac{b}{2a} \pm \sqrt{\left(\frac{c-d}{a} + \frac{b^2}{4a^2}\right)}$.

And consequently $x = \left(\frac{b}{2a} \pm \sqrt{\frac{4ac-4ad+b^2}{4a^2}}\right)^{\frac{2}{n}}$.

EXAMPLES FOR PRACTICE.

1. Given $x^2 - 8x + 10 = 19$ to find x. *Ans.* $x = 9$.
2. Given $x^2 - x - 40 = 170$ to find x. *Ans.* $x = 15$.
3. Given $3x^2 + 2x - 9 = 76$ to find x. *Ans.* $x = 5$.
4. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{3}{8} = 8$ to find x. *Ans.* $x = 1\frac{1}{2}$.
5. Given $2x^4 - x^2 = 496$ to find x. *Ans.* $x = 4$.
6. Given $\frac{1}{2}x - \frac{1}{3}\sqrt{x} = 22\frac{1}{6}$ to find x. *Ans.* $x = 49$.
7. Given $\frac{2}{3}x^2 + \frac{3}{4}x = \frac{4}{5}$ to find x. *Ans.* 6689.
8. Given $x^6 + 6x^3 = 2$ to find x.

   Ans. $x = \sqrt[3]{-3 \pm \sqrt{11}}$.
9. Given $x^2 + x = a$ to find x. *Ans.* $x = \sqrt{(a + \frac{1}{4})} - \frac{1}{2}$.
10. Given $x - \sqrt{x} = a$ to find x.

    Ans. $x = (\frac{1}{2} \pm \sqrt{a + \frac{1}{4}})^2$
11. Given $3x^{2n} - 2x^n = 25$ to find x.

    Ans. $x = (\frac{1}{3}\sqrt{76} + \frac{1}{3})^{\frac{1}{n}}$.
12. Given $\sqrt{1+x} - 2\sqrt[4]{1+x} = 4$ to find x.

    Ans. $x = (1 + \sqrt{5})^4 - 1$

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To find two numbers whoſe difference is 8, and product 240.

Let x *=to the leaſt number,*
Then will $x+8=$ *to the greater,*
And $x\times(x+8)=x^2+8x=240$ *by the queſtion ;*
Whence $x^2+8x+16=240+16=256$ *by completing the square.*
Alſo $x+4=\sqrt{256}=16$ *by evolution ;*
And therefore $x=16-4=12=$ *leſſer number, and* $12+8=20=$ *greater.*

2. To divide the number 60 into two ſuch parts that their product may be 864.

Let $x=$ *greater part,*
Then will $60-x=$ *less,*
And $x\times(60-x)=60x-x^2=864$ *by the queſtion ;*
That is $x^2-60x=-864$;
Whence $x^2-60x+900=-864+900=36$ *by completing the square.*
Alſo $x-30=\sqrt{36}=6$ *by extracting the root ;*
And therefore $x=6+30=36=$ *greater part,*
And $60-x=60-36=24$ *leſſer.*

3. Given the ſum of two numbers $=10(a)$, and the ſum of their ſquares $=58(b)$ to find thoſe numbers.

Let $x=$ *greater of the two numbers,*
Then will $a-x=$ *less.*
And $x^2+(a-x)^2=2x^2+a^2-2ax=b$ *by the queſtion,*
Or $x^2+\frac{a^2}{2}-ax=\frac{b}{2}$ *by diviſion.*

Or $x^2-ax=\frac{b}{2}-\frac{a^2}{2}=\frac{b-a^2}{2}$ *by transposition.*

Whence $x^2-ax+\frac{a^2}{4}=\frac{b-a^2}{2}+\frac{a^2}{4}=\frac{2b-a^2}{4}$ *by completing the square.*

Also, $x-\frac{a}{2}=\sqrt{\frac{2b-a^2}{4}}$ *by extracting the root;*

And therefore $x=\pm\sqrt{\frac{2b-a^2}{4}}+\frac{a}{2}=$ *greater number.*

And $a-\left(\frac{a}{2}\pm\sqrt{\frac{ab-a^2}{4}}\right)=\pm\sqrt{\frac{2b-a^2}{4}}+\frac{a}{2}=$ *less.*

Hence these two theorems, being put into numbers, give 7 and 3, for the numbers required.

4. Sold a piece of cloth for 24l. and gained as much *per cent.* as the cloth cost me; what was the price of the cloth?

Let $x=$ *pounds the cloth cost,*
Then $24-x=$ *whole gain.*
But $100 : x :: x : 24-x$ *by the question.*
Or $x^2=100\times(24-x)=2400-100x,$
That is $x^2+100x=2400,$
Whence $x^2+100x+2500=2400+2500=4900$ *by completing the square.*
And $x+50=\sqrt{4900}=70$ *by extraction of roots,*
Consequently $x=70-50=20$l. $=$ *price of the cloth.*

5. A person bought a number of oxen for 80l. and if he had bought four more for the same money, he would have paid 1l. less for each: how many did he buy?

Let the number of oxen be represented by x.
Then will $\frac{80}{x}$ *be the price of each.*

And $\frac{80}{x+4}$ = *price of each, if* $x+4$ *had cost* 80*l*.

But $\frac{80}{x} = \frac{80}{x+4} + 1$ *by the question.*

Or $80 = \frac{80x}{x+4} + x$,

Or $80x + 320 = 80x + x^2 + 4x$,

That is $x^2 + 4x = 320$,

Whence $x^2 + 4x + 4 = 320 + 4 = 324$ *by completing the square.*

And $x + 2 = \sqrt{324} = 18$ *by evolution,*

Consequently $x = 18 - 2 = 16 =$ *number of oxen required.*

6. What two numbers are those whose sum, product, and difference of their squares, are all equal to each other?

Let $x =$ *greater number,*

And $y =$ *less,*

Then $\left\{\begin{array}{l} x+y=xy \\ x+y=x^2-y^2 \end{array}\right\}$ *by the question,*

And $1 = \frac{x^2-y^2}{x+y} = x - y$, *or* $x = y + 1$ *from the 2d equation,*

Also $(y+1) + y = (y+1) \times y$ *from the first equation,*

Or $2y + 1 = y^2 + y$,

That is $y^2 - y = 1$,

Whence $y^2 - y + \frac{1}{4} = 1\frac{1}{4}$ *by completing the square,*

Also $y - \frac{1}{2} = \sqrt{1\frac{1}{4}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$ *by evolution,*

Consequently $y = \frac{\sqrt{5}}{2} + \frac{1}{2} = \frac{\sqrt{5}+1}{2}$, *and* $x = y + 1 = \frac{\sqrt{5}+3}{2}$.

And if these expressions be turned into numbers we shall have $x = 2.6180+$

and $y = 1.6180+$

7. There are four numbers in arithmetical progression, of which the product of the two extremes is 45, and that of the means 77; what are the numbers?

Let $x =$ *less extreme,*
and $y =$ *common difference.*

Then $x, x+y, x+2y, x+3y$, *will be the four numbers,*

$$\text{and } \left\{ \begin{array}{l} x \times (x+3y) = x^2 + 3xy = 45 \\ (x+y) \times (x+2y) = x^2 + 3xy + 2y^2 = 77 \end{array} \right\} \text{ by the question,}$$

Whence $2y^2 = 77 - 45 = 32$ *by subtraction, and* $y^2 = \frac{32}{2} = 16$ *by division.*

Or $y = \sqrt{16} = 4$ *by evolution.*

Therefore $x^2 + 3xy = x^2 + 12x = 45$ *by the 1st equation.*

Also $x^2 + 12x + 36 = 45 + 36 = 81$ *by completing the square,*

And $x + 6 = \sqrt{81} = 9$ *by the extraction of roots.*

Consequently $x = 9 - 6 = 3$, *and the numbers are* 3, 7, 11 *and* 15.

8. To find 3 numbers in geometrical progression, whose sum shall be 14, and the sum of their squares 84.

Let x, y, *and* z *be the numbers sought,*

Then $xz = y^2$ *by the nature of proportion,*

$$\text{And } \left\{ \begin{array}{l} x + y + z = 14 \\ x^2 + y^2 + z^2 = 84 \end{array} \right\} \text{ by the question,}$$

But $x + z = 14 - y$ *by the 2d equation.*

And $x^2 + 2xz + z^2 = 196 - 28y + y^2$ *by squaring both sides,*

Or $x^2 + z^2 + 2y^2 = 196 - 28y + y^2$ *by putting* $2y^2$ *for its equal* $2xz$.

That is $x^2 + z^2 + y^2 = 196 - 28y$ *by subtraction,*

Or $196 - 28y = 84$ *by equality,*

Hence $y = \frac{196 - 84}{28} = 4$ *by transposition and division.*

Again, $xz=y^2=16$, *or* $x=\frac{16}{z}$ *by the* 1*st equation.*

And [illegible] $\frac{16}{z}+4+z=14$ *by the* 2*d equation,*

Or $16+4z+z^2=14z$, *or* $z^2-10z=-16$,

Whence $z^2-10z+25=25-16=9$ *by completing the square,*

And $z-5=\sqrt{9}=3$, *or* $z=3+5=8$,

Consequently $x=14-y-z=14-4-8=2$, *and the numbers are* 2, 4, 8.

9. The sum (s) and the product (p) of any two numbers being given; to find the sum of the squares, cubes, biquadrates, &c. of those numbers.

Let the two numbers be denoted by x *and* y.

Then will $\left\{\begin{array}{l} x+y=s \\ xy=p \end{array}\right\}$ *by the question.*

But $(x+y)^2=x^2+2xy+y^2=s^2$ *by involution,*

and $x^2+2xy+y^2-2xy=s^2-2p$ *by subtraction.*

That is $x^2+y^2=s^2-2p=$ *sum of the squares.*

Again, $(x^2+y^2)\times(x+y)=(s^2-2p)\times s$ *by multiplication,*

Or $x^3+xy\times(x+y)+y^3=s^3-2sp$,

Or $x^3+sp+y^3=s^3-2sp$ *by substituting* sp *for its equal* $xy\times(x+y)$;

And therefore $x^3+y^3=s^3-3sp=$ *sum of the cubes.*

In like manner $(x^3+y^3)\times(x+y)=(s^3-3sp)\times s$ *by multiplication,*

Or $x^4+xy\times(x^2+y^2)+y^4=s^4-3s^2p$,

Or $x^4+p\times(s^2-2p)+y^4=s^4-3s^2p$, *by substituting* $p\times(s^2-2p)$ *for its equal* $xy\times(x^2+y^2)$;

And consequently, $x^4+y^4=s^4-3s^2p-p\times(s^2-2p)=s^4-4s^2p+2p^2=$ *sum of the biquadrates, or fourth powers;*

And the sum of the nth. powers is $s^n-ns^{n-2}p+n.\frac{n-3}{2}.s^{n-4}p^2-n.\frac{n-4}{2}.\frac{n-5}{3}.s^{n-6}p^3+n.\frac{n-5}{2}.\frac{n-6}{3}.\frac{n-7}{4}.s^{n-8}p^4$, *&c.*

10 The sum (a) and the sum of the squares (b) of four numbers in geometrical progression being given; to find those numbers.

means

Let x and y denote the two [redacted],

Then will $\frac{x^2}{y}$ and $\frac{y^2}{x}$ be the two extremes by the nature of proportion.

Also, let the sum of the two means $=s$, and their product $=p$.

Then will the sum of the two extremes $=a-s$ by the question, and their product $=p$, by the nature of proportion.

Hence $\left\{\begin{array}{l} x^2+y^2=s^2-2p \\ \dfrac{x^4}{y^2}+\dfrac{y^4}{x^2}=(a-s)^2-2p \end{array}\right\}$ *by the last problem.*

And $x^2+y^2+\frac{x^4}{y^2}+\frac{y^4}{x^2}=s^2+(a-s)^2-4p=b$ *by the question.*

Again, $\frac{x^2}{y}+\frac{y^2}{x}=a-s$ *by the question,*

Or $x^3+y^3=xy\times(a-s)=p\times(a-s)$.

But $x^3+y^3=s^3-3sp$ *by the last problem;*

And therefore $p\times(a-s)=s^3-3sp$ *by equality.*

Or $pa-ps+3ps=pa+2ps=s^3$

Or $p=\frac{s^3}{a+2s}$ *by division.*

Whence $s^2+(a-s)^2-4p=s^2+(a-s)^2-\frac{4s^3}{a+2s}=b$, *by substitution.*

Or $s^2+\frac{b}{a}s=\frac{a^2-b}{2}$ *by reduction.*

And $s=\sqrt{\left(\frac{a^2-b}{2}+\frac{b^2}{4a^2}\right)}-\frac{b}{2a}$ *by comp. the square, and extracting the root.*

And from this value of s, all the rest of the quantities p, x, and y, may be readily determined.

QUESTIONS FOR PRACTICE.

1. What two numbers are thoſe whoſe ſum is 20, and their product 36? *Anſ.* 2 and 18.

2. To divide the number 60 into two ſuch parts, that their product may be to the ſum of their ſquares in the ratio of 2 to 5. *Anſ.* 20 *and* 40.

3. The difference of two numbers is 3, and the difference of their cubes is 117; what are thoſe numbers? *Anſ.* 2 *and* 5.

4. A company at a tavern had 8l. 15s. to pay for their reckoning; but, before the bill was ſettled, two of them left the room, and then thoſe who remained had 10s. a-piece more to pay than before: how many were there in company? *Anſ.* 7.

5. A grazier bought as many ſheep as coſt him 60l. and, after reſerving 15 out of the number, he ſold the remainder for 54l. and gained 2s. a head by them; how many ſheep did he buy? *Anſ.* 75.

6. There are two numbers whoſe difference is 15, and half their product is equal to the cube of the leſſer number; what are thoſe numbers? *Anſ.* 3 *and* 18.

7. A perſon bought cloth for 33l. 15s. which he ſold again at 2l. 8s. *per* piece, and gained by the bargain as much as one piece coſt him; required the number of pieces? *Anſ.* 15.

8. What number is that, which when divided by the product of its two digits, the quotient is 3; and if 18 be added to it, the digits will be inverted? *Anſ.* 24.

9. What two numbers are thoſe whoſe ſum multiplied by the greater is equal to 77; and whoſe difference multiplied by the leſſer is equal to 12? *Anſ.* 4 *and* 7.

10. To find a number such that if you subtract it from 10, and multiply the remainder by the number itself, the product shall be 21. *Ans.* 7 *or* 3.

11. To divide 100 into two such parts, that the sum of their square roots may be 14.

Ans. 64 *and* 36.

12. It is required to divide the number 24 into two such parts, that their product may be equal to 35 times their difference. *Ans.* 10 *and* 14.

13. The sum of two numbers is 8, and the sum of their cubes is 152; what are the numbers?

Ans. 3 *and* 5.

14. The sum of two numbers is 7, and the sum of their 4th powers is 641; what are the numbers?

Ans. 2 *and* 5.

15. The sum of two numbers is 6, and the sum of their 5th powers is 1056; what are the numbers?

Ans. 2 *and* 4.

16. The sum of four numbers in arithmetical progression is 56, and the sum of their squares is 864; what are the numbers? *Ans.* 8, 12, 16, and 20.

17. To find four numbers in geometrical progression whose sum is 15, and the sum of their squares 85?

Ans. 1, 2, 4, *and* 8.

18. It is required to find four numbers in arithmetical progression, such that their common difference may be 4, and their continued product 176985.

Ans. 15, 19, 23, *and* 27.

19. Two partners, A and B, gained 140l. by trade; A's money was three months in trade, and his gain was 60l. less than his stock; and B's money, which was 50l. more than A's, was in trade 5 months; what was A's stock. *Ans.* 100l.

OF THE NATURE AND FORMATION OF

EQUATIONS IN GENERAL.

All equations of ſuperior orders are generated by the multiplication of thoſe of inferior orders, involving the ſame unknown quantity.

Thus, *a quadratic equation* is formed by the multiplication of two ſimple equations.

A cubic equation is produced by the continued multiplication of three ſimple equations, or by one quadratic and one ſimple equation.

A biquadratic equation is generated by the continued multiplication of four ſimple equations; or by two quadratic equations; or by one cubic and one ſimple equation, &c.

For, ſuppoſe the unknown quantity to be x, and its values in any ſimple equation to be a, b, c, d, &c.

Then thoſe ſimple equations, by bringing all the terms to one ſide, will become $x-a=o$, $x-b=o$, $x-c=o$, $x-d=o$, &c.

And the product of any two of theſe, as $(x-a)\times(x-b)=o$, will form a *quadratic equation*, or one of two dimenſions.

The product of any three of them, as $(x-a)\times(x-b)\times(x-c)=o$, will form a *cubic equation*, or one of three dimenſions.

The product of any four of them, as $(x-a)\times(x-b)\times(x-c)\times(x-d)=o$, will form a biquadratic equation, or one of four dimenſions, &c.

From hence it appears, that every equation has as many roots as it has ſimple equations which produce

it, or as there are units in the higheſt dimenſion of the unknown quantity.

For if any of the values of x (a, b, c, d,) be ſubſtituted in the place of x in the biquadratic equation $(x-a)\times(x-b)\times(x-c)\times(x-d)$, all the terms of that equation will vaniſh, and the whole will be equal to nothing.

And as there are no other quantities, beſides theſe four, which, ſubſtituted in the place of x, will make the product vaniſh, it is plain that the equation cannot poſſibly have more than four roots, or admit of more than four ſolutions.

After the ſame manner, it may alſo be ſhewn that no equation whatever can have more roots than it contains dimenſions of the unknown quantity.

To make this ſtill plainer by an example in numbers: ſuppoſe the equation to be reſolved be $x^4-10x^3+35x^2-50x+24=0$, and that you diſcover this equation to be the ſame with the product of $(x-1)\times(x-2)\times(x-3)\times(x-4)$

Then it may be inferred that the four values of x are 1, 2. 3, and 4; for any of theſe numbers being put for x will make that product, and conſequently $x^4-10x^3+35x^2-50x+24$ equal to nothing, as it is in the propoſed equation.

And it is certain that there can be no other values of x beſides theſe four; ſince, if any other number be ſubſtituted for x in thoſe factors, none of them will vaniſh, and therefore their product cannot be equal to nothing, as it ought to be by the equation.

The roots of equations are, alſo, either *poſitive* or *negative*, according as the roots of the ſimple equation from whence they are produced are poſitive or negative.

Thus if you ſuppoſe $x=-a$, $x=b$, $x=-c$, and $x=d$, then will $x+a=0$, $x-b=0$, $x+c=0$ and $x-d$

$=o$, and the equation $(x+a)\times(x-b)\times(x+c)\times(x-d)$ $=o$ will have its roots $-a$, $+b$, $-c$, $+d$.

But the *signs* and *co-efficients* of equations will be best understood by considering the following table; where the simple equations $x-a$, $x-b$, &c. being multiplied continually together, produce, successively, the higher equations.

$$x-a=o$$
$$x-b=o$$

$$\left.\begin{array}{l} x^2-ax \\ \quad -bx+ba \end{array}\right\} =o, \textit{ a quadratic,}$$

$$\times\, x-c=o$$

$$\left.\begin{array}{r} x^3-a \\ -b \\ -c \end{array}\right\} x^2 \left.\begin{array}{r} +ab \\ +ac \\ bc \end{array}\right\} x-abc=o, \textit{ a cubic,}$$

$$\times\, x-d=o$$

$$\left.\begin{array}{r} x^4-a \\ -b \\ -c \\ -d \end{array}\right\} x^3 \left.\begin{array}{r} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{array}\right\} x^2 \left.\begin{array}{r} -abc \\ -abd \\ -acd \\ -bcd \end{array}\right\} x+abcd=o, \textit{ a biquadratic,}$$

&c.

From the inspection of these equations it is plain, that the co-efficient of the first terms is *unity*.

The co-efficient of the third term is *the sum of all the roots* (a, b, c, d) *with contrary signs.*

The co-efficient of the third term is *equal to the sum of the rectangles of the roots, or of all the products that can possibly arise by combining them two and two.*

The co-efficient of the fourth term is *equal to the sum of all the products that can possibly arise by combining them, three by three*; and so on for any other co-efficient whatever.

The last term *is always equal to the product of all the roots with contrary signs*; and this reasoning will hold, whether the roots be positive or negative.

It likewise appears, from inspection, that the signs of all the terms of any equation in the table are alternately $+$ and $-$.

Thus the first term is always some pure power of x, and is positive.

The second term is some power of x, multiplied by the quantities $-a$, $-b$, $-c$, &c. and since these quantities are all negative, it follows that the term itself must be negative also.

The third term has the product of any two of these quantities $(-a, -b, -c,)$ for its co-efficient, and is therefore positive; since $-\times-$, as well as $+\times+$, gives, $+$, or an affirmative quantity.

For the same reason, the next co-efficient, which is formed of the products of any three of these negative quantities, must be negative; and the next following, being made up of the products of any four of the said negative quantities, must be positive; and so on.

And, from this reasoning, it plainly appears that *when all the roots are positive, the signs are plus and minus alternately*.

But if the roots be all negative, as $x=-a$, $x=-b$, $x=-c$, $x=-d$, then $(x+a)\times(x+b)\times(x+c)\times(x+d)=0$, will express the equation to be produced; and all the terms will plainly be positive.

So that, *when all the roots of an equation are negative it is plain that there can be no change in the signs of the terms of that equation.*

And, in general, there will be as many positive roots in any equation, as there are changes in the

ſigns of the terms of that equation, from + to —, or from — to +; and all the reſt of the roots will be negative.

From this rule it follows that, in quadratic equations, the two roots may be either both poſitive, or both negative, or one negative and one poſitive.

Thus, in the equation, $x^2 \begin{matrix}-ax\\-bx\end{matrix}+ab=o$, or $(x-a)\times(x-b)$ there are two changes of the ſigns, and therefore the roots are both poſitive.

In the equation $x^2 \begin{matrix}+ax\\+bx\end{matrix}+ab=o$, or $(x+a)\times(x+b)$ there is no change of the ſigns, and conſequently they are both negative.

And, in the equation $x^2 \begin{matrix}-ax\\-bx\end{matrix}-ab=o$, or $(a-a)\times(x+b)$ one of the roots will be affirmative and one negative; for, as the firſt term is poſitive and the laſt negative, there can be but one change in the ſigns, whether the ſecond term be + or —.

In cubic equations, the roots may be all poſitive, or all negative; or two of them may be negative and one poſitive, or one negative and two poſitive.

Thus, in the equation $(x-a)\times(x-b)\times(x-c)=o$ the ſigns will be alternately + and —; and, as the number of changes is three, the roots muſt be all poſitive.

In the equation $(x+a)\times(x+b)\times(x+c)=o$, where there are no changes of the ſigns the roots muſt be all negative.

In the equation $x^3-(a-b+c)x^2+(ab-ac-bc)x+abc=o$, or $(x-a)\times(x-b)\times(x+c)$ the number of changes will be two, and conſequently two of the roots will be poſitive and one negative.

For if $a+b$ be greater than c, the ſecond term muſt be negative, its co-efficient being $-a$, $-b$, $+c$;

and if $a+b$ be lefs than c, the third term muft be negative, its co-efficient $+ab-ac-bc$ being in that cafe negative.

In the equation $x^3+(a+b-c)x^2+(ab-ac-bc)x-abc=0$, there can be only one change of the figns, and therefore one of the roots is pofitive, and the other two negative.

For if $a+b$ be lefs than c, then the fecond term is negative, and the third muft be negative alfo: and if $a+b$ be greater than c, the fecond term will be pofitive, and there can be but one change in the other two terms, whatever may be their figns.

And, in the fame manner, this reafoning may be extended to equations of higher dimenfions, and therefore the rule will be found to be true in all kinds of equations whatever.

PROBLEM I.

*To increase or diminish the roots of an equation by any given quantity.**

RULE.

1. Take fome new letter, and connect it with the given quantity by the figns $-$ or $+$, according as it is required to be increafed or diminifhed.

2. Subftitute the powers of this quantity in the equation, inftead of the powers of the unknown letter, and there will arife a new equation, whofe roots will be augmented or diminifhed as required.

* When a cubic equation has two equal roots, it may always be reduced to a lower dimenfion, and the folution, by that means, made more eafy.

EXAMPLES.

1. Let the quadratic equation $x^2+8x+15=0$, be given; it is required to increase its roots by 7.

Suppose $x=y-7$,

$$\begin{aligned} \textit{Then } x^2 &= y^2-14y+49 \\ 8x &= +\ 8y-56 \\ 15 &= \qquad +15 \\ \hline & y^2-6y+8=0= \textit{ to the equation required.}^* \end{aligned}$$

2. Let $x^3-px^2+qx-r=0$, be the equation given; it is required to diminish the roots by the quantity e.

Suppose $x=y+e$;

$$\left.\begin{aligned} \textit{Then } x^3 &= y^3+3ey^2+3e^2y+e^3 \\ -px^2 &= -py^2-2pey-pe^2 \\ +qx &= \qquad +qy+qe \\ -r &= \qquad\qquad -r \end{aligned}\right\} =0, \textit{ the new equation required.}\dagger$$

3. Let $x^3+x^2-10x+8=0$, be given, and let its roots be increased by 4.

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* For in the former equation $x^2+8x+15=0$, the roots are -3, and -5, and in the equation $y^2-6y+8=0$, the roots are 2 and 4 therefore the difference is 7, as was required.

† The last term of this transformed equation is the same as the given equation, having e in place of y.

And from this it appears that, if the last term of any equation is to be destroyed, the difficulty will be no less than that of solving the original equation itself.

Suppose $x=y-4$;
Then $x^3=y^3-12y^2+48y-64$
$+x^2=+y^2-8y+16$
$-10x=-10y+40$
$+8=+8$

Sum $=y^3-11y^2+30y=0$,
Or $y^2-11y+30y=0$, *the equation required.**

In which equation y is found $=6$, *and consequently* $x=2$.

PROBLEM II.

To take away the second term from any equation.

RULE.

1. Divide the co-efficient of the second term by the index of the higheſt power of the unknown quantity.

2. Annex the quotient, with its ſign changed, to ſome new letter, and this, being ſubſtituted for its equal in the given equation, will deſtroy the ſecond term, as required.

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* In this example the given equation is reduced to a quadratic; and in the preſent caſe, as well as in all others, where the laſt term vaniſhes, the number aſſumed -4 is one of the roots of the propoſed equation.

The affirmative roots of an equation are changed into negative ones of the ſame value, and the negative roots into affirmative ones, by only changing the ſigns of the terms alternately, beginning with the ſecond.

EXAMPLES.

1. Let the quadratic equation $x^2-8x+15=0$ be given: it is required to take away its second term.

Suppose $x=y+4\left(y+\frac{8}{2}\right)$;

Then $x^2=y^2+8y+16$

$-8x=\quad -8y-32$

$+15=\qquad +15$

$y^2-1=0=$ *equation required.* *

2. Let the equation $x^3-9x^2+26x-34=0$ be given; it is required to exterminate its second term.

Suppose $x=y+3\left(y+\frac{9}{3}\right)$;

Then $x^3=y^3+9y^2+27y+27$

$-9x^2=\quad -9y^2-54y-81$

$+26x=\qquad +26y+78$

$-34=\qquad\qquad -34$

$y^3-y-10=0=$ *equation required.*

Thus the roots of the equation $x^4-x^3-19x^2+49x-30=0$, are $+1$, $+2$, $+3$, -5; but by changing only the second and fourth terms, the equation becomes $x^4+x^3-19x^2-49x-30=0$, and the roots are -1, -2, -3, $+5$.

All the roots of an equation may also be made affirmative or negative, by increasing or diminishing each of them by some known quantity.

* From this example it appears, that any quadratic equation may be solved without completing the square, by only taking away the second term; for since $y^2=1$, or $y=\sqrt{1}=1$, we shall have $x=y+4=1+4=5$, the root required. And the same may be shewn of any other affected quadratic equation whatever.

3. Let $x^4+8x^3-5x^2+10x-4=0$ be given, to exterminate the second term.*

Suppose $x=y-2\left(y-\frac{8}{4}\right)$;

$$\begin{array}{rl}
\textit{Then } x^4 = & y^4-8y^3+24y^2-32y+16 \\
+8x^3 = & \phantom{y^4-{}}8y^3-48y^2+96y-64 \\
-5x^2 = & \phantom{y^4-8y^3+{}}-5y^2+20y-20 \\
+10x = & \phantom{y^4-8y^3+24y^2-{}}+10y-20 \\
-4 = & \phantom{y^4-8y^3+24y^2-32y+{}}-4 \\
\hline
 & y^4-29y^2+94y-92=0= \textit{equation required.}
\end{array}$$

4. Let $x^4-px^3+qx^2-rx+s=0$ be given, to exterminate the second term.

Suppose $x=y+\frac{p}{4}$;

$$\begin{array}{rl}
\textit{Then } x^4 = & y^4+py^3+\frac{3p^2y^2}{8}+\frac{p^3y}{16}+\frac{p^4}{256} \\
-px^3 = & -py^3-\frac{3p^2y^2}{4}-\frac{3p^3y}{16}-\frac{p^4}{64} \\
+qx^2 = & +qy^2+\frac{pqy}{2}+\frac{qp^2}{16} \\
-rx = & -ry-\frac{rp}{4} \\
+s = & +s \\
\hline
 & \\
\hline
\end{array}$$

---

* Since the sum of all the roots, in any equation, are equal to the co-efficient of the second term it follows that, when the second term is wanting, the equation has both affirmative and negative roots, and that the sum of the affirmative roots is equal to the sum of the negative ones.

Thus in the cubic equation x^3-7x-6, the three roots are $+3$, -2, and -1, where it is evident that $3=2+1$.

PROBLEM III.

To find whether some or all the roots of an equation be rational; and, if so, what they are.

RULE.*

1. Find all the divifors of the laft term, and fubftitute them one by one for the unknown quantity.

2. Then if the pofitive and negative terms deftroy each other, the divifor, fo fubftituted, will be one of the roots of the equation.

3. But if none of the divifors fucceed, the roots are, for the general part, either irrational or impoffible.

Note. When the divifors of the laft term are too numerous, they may be diminifhed by changing the equation into another whofe roots are augmented or decreafed by an unit, or fome other known quantity.

EXAMPLES.

1. Let $x^3-4x^2-7x+10=0$ be the equation propofed.

Then the divisors of (10) *the laft term will be* $+1$, -1 $+2$, -2, $+5$, -5, $+10$, -10.

---

* Since the laft term, in any equation, is always equal to the product of all the roots in that equation, thofe roots muft, therefore, neceffarily be found in the number of its divifors.

But this, it is evident, can hold only when the roots are commenfurate, or whole numbers.

And these being substituted successively instead of x, will give.

$$1-4-7+10=0$$
$$-1-4+7+10=12$$
$$8-16-14+10=-12$$
$$-8-16+14+10=0$$
$$125-100-35+10=0$$

Therefore $+1$, -2, *and* $+5$ *are the three roots of the equation required.*

2. Let $y^4-4y^3-8y+32=0$ be the equation proposed.

1. *Change it into another, the number of whose divisors shall be less; thus,*

Suppose $y=x+1$

Then $y^4=x^4+4x^3+6x^2+4x+1$

$$-4y^3=-4x^3-12x^2-12x-4$$
$$-8y=-8x-8$$
$$+32=+32$$

$$x^4-6x^2-16x+21=0= \textit{new equation.*}$$

2. *The divisors of the last term (21) of this new equation are*

$$1, -1, +3, -3, +7, -7, +21, -21.$$

And if these be substituted successively instead of x, we shall have.

$$1-6-16+21=0$$
$$1-6+16+21=32$$
$$81-54-48+21=0$$
$$81-54+48+21=96$$

&c. where none of the others succeed.

So that 1 and 3 are the only rational roots, the other two being impossible.

* *Note.* The divisor of the last term of this new equation may be diminished in the same manner as before.

3. Let $x^3+3ax^2-4a^2x-12a^3=0$, be the equation proposed.

Here the numeral divisors of the last term ($12a^3$) *are*

$$1,\ -1,\ +2,\ -2,\ +3,\ -3,\ +4,\ -4,\ +6,\ -6,\ +12,\ -12.$$

And by substituting these successively instead of x, *we shall have*

$$\begin{array}{r}
1+3-4-12=-12\\
-1+3+4-12=-6\\
8+12-8-12=0\\
-8+12+8-12=0\\
27+27-12-12=30\\
-27+27+12-12=0
\end{array}$$

Therefore the three roots are $2a$, $-2a$, *and* $-3a$.

PROBLEM IV.

To discover the roots of equations by Sir Isaac Newton's *method of divisors.*

RULE.

1. Instead of the unknown quantity, substitute successively three, or more, terms of the arithmetical progression 2, 1, 0, —1, —2.

2. Collect all the terms of the equation into one sum, and place them, together with their divisors, in perpendicular lines, right against the corresponding terms of the progression 2, 1, 0, —1, —2.

3. Seek amongst the divisors for an arithmetical progression, whose terms correspond with the order of the terms 2, 1, 0, —1, —2, and whose common difference is either unit, or some divisor of the co-efficient of the

higheſt power of the unknown quantity in the given equation.

4. Divide that term of the progreſſion, thus found, which ſtands againſt the term o, in the firſt progreſſion, by the ratio or common difference.

5. To the quotient laſt found, prefix the ſign + or —, according as the progreſſion is increaſing or decreaſing and this number being ſubſtituted for the unknown quantity, will be found to be one of the roots of the equation.

Note. When there is more than one progreſſion, the roots muſt be taken out of each.

EXAMPLES.

1. Let $x^3-x^2-10x+6=0$ be the equation propoſed.

Then by substituting successively the terms of the progression 2, 1, 0,—1 *instead of* x, *the work will stand as follows:*

1ſt prog.	*results.*	*divisors.*	*2d prog.*
2	—10	1.2.5.10	5
1	— 4	1.2 4.	4
0	+ 6	1.2.3. 6	3
—1	+14	1.2.7.14	2

And —3, *the term standing against* 0, *being substituted for* x, *gives*, $-27-9+30+6=0$, *and therefore* —3 *is a root of the equation.*

2. Let $2x^3-5x^2+4x-10=0$ be the equation propoſed.

Then, by substituting successively the terms of the progression, 2, 1, 0, —1, —2, *instead of* x, *the work will stand as follows:*

1st prog.	*results.*	*divisors.*	*2d prog.*
2	—6	1.2.3. 6	1
1	—9	1.3.9.	3
0	—10	1.2.5.10	5
—1	—21	1.3.7.21	7
—2	—54	1.2.3. 6.9	9

Here 5, the term standing against 0, being divided by 2, the common difference, gives $2\frac{1}{2}$ *; and this being substituted for x, gives* $31\frac{1}{4}-31\frac{1}{4}+10-10=0$; *and therefore* $2\frac{1}{2}$ *is a root of the equation.*

3. Let $x^4+x^3-29x^2-9x+180=0$, be the equation proposed.*

Then by substituting as before, the work will stand as follows:

1st pro.	*results.*	*divisors.*	*progressions.*			
2	70	1.2.5.7.10.14 &c	1	2	5	7
1	144	1.2.3.4. 6. 8 &c.	2	3	4	6
0	180	1.2.3.4. 5. 6 &c.	3	4	3	5
—1	160	1.2.4.5. 8.10 &c.	4	5	2	4
—2	90	1.2.3.5. 6. 9 &c.	5	6	1	3

So that here are four progressions, and the numbers 3, 4, —3, and —5, being each substituted for x, make the whole equation vanish, and are therefore the roots required.

4. Given $x^3-x^2-46x-72=0$, to find the values of x, by the method of divisors. *Ans.* $+9$, -2 *and* -4.

5. Given $x^4-4x^3-19x^2+46x+120=0$, to find the values of x, by the method of divisors.

Ans. $+5$, $+4$, -3, -2.

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* Several other rules for discovering the roots of equations may be found in Newton's and Maclaurin's Algebra.

PROBLEM V.

To find the roots of cubic equations, according to the method of CARDON.

RULE.*

1. Take away the second term of the equation, by problem 2d, and it will be reduced to this form $x^3 \pm ax = \pm b$.

2. Substitute the values of a and b, with their proper signs, in the following expression, and it will give the root required. Thus:

$$x=\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}}=\text{root}$$

required.

---

* The rule from whence this method is derived, is $x=\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}+\sqrt[3]{\tfrac{1}{2}b-\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}$; and the investigation of it is as follows:

Let the equation, whose root is required, be $x^3+ax=b$.
And assume $y+z=x$, and $3yz=-a$.

Then, by substituting these values in the given equation, we shall have, $y^3+3y^2z+3yz^2+z^3+a\times(y+z)=y^3+z^3+3yz\times(y+z)+a\times(y+z)=y^3+z^3-a\times(y+z)+a\times(y+z)=y^3+z^3=b$.

And, if, from the square of this last equation, there be taken 4 times the cube of the equation $yz=-\tfrac{1}{3}a$, we shall have $y^6-2y^3z^3+z^6=b^2+\tfrac{4}{27}a^3$, or $y^3-z^3=\sqrt{(b^2+\tfrac{4}{27}a^3}$.

But the sum of this equation and $y^3+z^3=b$, is $2y^3=b+\sqrt{(b^2+\tfrac{4}{27}a^3)}$ and their difference is $2z^3=b-\sqrt{(b^2+a^3\,\tfrac{4}{27})}$; whence y is found $=\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}$ and $z=\sqrt[3]{\tfrac{1}{2}b-\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}$.

*Note.** When a is negative, and $\frac{1}{27}a^3$ is greater than $\frac{1}{4}b^2$, the folution, by this rule, cannot be generally obtained.

EXAMPLES.

1. Let $y^3+3y^2+9y=13$, be the equation propofed; it is required to find the value of y.

1. *In order to deftroy the fecond term, let* $y=x-1$; *then*

$$\begin{aligned} y^3 &= x^3-3x^2+3x-1 \\ 3y^2 &= \quad\;\; +3x^2-6x+3 \\ 9y &= \qquad\qquad\;\; +9x-9 \\ \hline & \quad x^3+6x-7=13 \\ or & \quad x^3+6x=20. \end{aligned}$$

And from hence it appears that $y+z$, or its equal x, is $=\sqrt[3]{\frac{1}{2}b+(\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3})}+\sqrt[3]{\frac{1}{2}b-\sqrt{(\frac{1}{4}b^2+\frac{1}{27}a^3)}}$, which is the theorem.

Or, fince z is equal $-\frac{a}{3y}$, it will be $y+z=y-\frac{a}{3y}=x=$

$$\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}-\frac{\frac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b\times\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}},$$ *the fame as the rule.*

This method of folving cubic equations is ufually afcribed to *Cardon*; but the invention is not his—The authors of it were *Scipio Ferreus*, and *Nicholas Tartalea*, who difcovered it about the fame time, independently of each other, as is proved by *M. de Montucla*, in his *Hiftoire des Mathématiques*.

* This is called the irreducible cafe; and, notwithftanding feveral of the moft eminent mathematicians in Europe have attempted the folution of it, no general rule has yet been difcovered.

The ufual method is by a table of fines, or by throwing the expreffion into an infinite feries, and finding the fum of a certain number of terms according to the degree of exactnefs required.

2. *For a put 6, and for b 20, and we shall have*
$$x=\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b^2+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}}=$$
$$\sqrt[3]{10+\sqrt{(100+8)}}-\frac{2}{\sqrt[3]{10+\sqrt{(100+8}}}=$$
$$\sqrt[3]{10+10.3923}-\frac{2}{\sqrt[3]{10+10.3923}}=\sqrt[3]{20.3923}-$$
$$\frac{2}{\sqrt[3]{20.3923}}=2.732-\frac{2}{2.732}=2.732-{}'732=2;$$ *that is* $x=2$, *and consequently* $y=1=$ *root required.*

2. Given $x^3-6x=-9$, to find the value of x.
Here $a=-6$, *and* $b=-9$; *and therefore we shall have*
$$x=\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3}}}=$$
$$\sqrt[3]{-4\tfrac{1}{2}+\sqrt{(20\tfrac{1}{4}-8)}}-\frac{-2}{\sqrt[3]{-4\tfrac{1}{2}+\sqrt{(20\tfrac{1}{4}-8)}}}=$$
$$\sqrt[3]{-4\tfrac{1}{2}+3\tfrac{1}{2}}-\frac{-2}{\sqrt[3]{-4\tfrac{1}{2}+3\tfrac{1}{2}}}=\sqrt[3]{-1}-\frac{-2}{\sqrt[3]{-1}}=-1-$$
$$\frac{-2}{-1}=-1-2=-3;$$ *that is* $x=-3=$ *root required.*

EXAMPLES FOR PRACTICE.

1. Given $x^3-6x^2+10x=8$, to find x. *Ans.* $x=4$.
2. Given $y^3+30y=117$, to determine y. *Ans.* $y=3$.
3. Given $y^3-36y=91$, to determine y. *Ans.* $y=7$.
4. Given $y^3-3y=18$ to determine y. *Ans.* $y=3$.
5. Given $y^3+24y=250$, required y. *Ans.* $y=5.05$.
6. Given $y^6-3y^4-2y^2-8=0$, to find y. *Ans.* $y=2$.

PROBLEM VI.

To find the roots of biquadratic equations, according to the method of DES CARTES.

RULE.*

1. Take away the second term of the equation by problem 2, and it will be reduced to the form $x^4+qx^2+rx+s=0$.

2. From the cubic equation $y^3+2qy^2+(q^2-4s)y-r^2=0$, take the second term, and find the value of y by the last problem.

3. Let e be assumed $=\sqrt{y}$, $f=\frac{1}{2}q+\frac{1}{2}e^2-\frac{r}{2e}$, and $g=\frac{1}{2}q+\frac{1}{2}e^2+\frac{r}{2e}$.

4. Find the roots of the two quadratic equations $x^2+ex+f=0$, and $x^2-ex+g=0$, and they will be the four roots of the biquadratic required.

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* *Investigation of the rule.* Let the given equation $x^4+qx^2+rx+s=0$, be equal to the product of the two quadratic equations $x^2+ex+f=0$, and $x^2-ex+g=0=x^4+(f+g-e^2)x^2+(eg-ef)x+fg=0$.

Then, by equating the homologous terms, we shall have $f+g-e^2=q$, $eg-ef=r$, and $fg=s$; and therefore $f=\frac{1}{2}q+\frac{1}{2}e^2-\frac{r}{2e}$, $g=\frac{1}{2}q+\frac{1}{2}e^2+\frac{r}{2e}$, and $s=f\times g=\frac{1}{4}q^2+\frac{1}{2}qe^2+\frac{1}{4}e^4-\frac{r^2}{4e^2}$.

And from this last equation we shall have $e^6+2qe^4+(q^2-4s)\times e^2=r^2=$ to a cubic equation, in which the value of e may be found, as in the last problem.

EXAMPLES.

1. Let $z^4-4z^3-8z+32=0$, be the equation proposed, in which it is required to find the value of z.

1. *To take away the second term, let* $x+1=z$; *then*

$$\begin{array}{rl} z^4= & x^4+4x^3+\ 6x^2+\ 4x+\ 1 \\ -4z^3= & \quad -4x^3-12x^2-12x-\ 4 \\ -8z= & \qquad\qquad\qquad -\ 8x-\ 8 \\ +32= & \qquad\qquad\qquad\qquad\quad +32 \\ \hline & x^4-6x^2-16x+21=0, \text{ or} \end{array}$$

$y^3-12y^2-48y-256=0$, *for the cubic equation.*

2. *To take away the second term from this equation, let* $p+4=y$; *then*

$$\begin{array}{rl} y^3= & p^3+12p^2+48p+\ 64 \\ -12y^2= & \quad -12p^2-26p-192 \\ -41y= & \qquad\qquad -48p-192 \\ -256= & \qquad\qquad\qquad -256 \\ \hline & p^3-96p=576 \end{array}$$

But $f(=\frac{1}{2}q+\frac{1}{2}e^2-\frac{r}{2e}$ and $g\ (=\frac{1}{2}q+\frac{1}{2}e\ +\frac{r}{2e})$ are also known; and therefore the roots of the quadratic equations $x^2+ex+f=0$, and $x^2-ex+g=0$ may be determined, and are the four roots of the biquadratic equation required. Q. E. I.

Note. The co-efficient of x is put equal to e, in both the equations, because, when the second term as wanting, the sum of the positive roots is always equal to the sum of the negative ones, with a contrary sign.

This rule has sometimes been ascribed to *Des Cartes*, and sometimes to *Bombelli*, an Italian; but the original inventor of it was *Louis Ferrari*.

3. *To find the value of p, by* CARDAN'S *rule for cubic equations.*

$$\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}-\frac{\tfrac{1}{3}a}{\sqrt[3]{\tfrac{1}{2}b+\sqrt{(\tfrac{1}{4}b^2+\tfrac{1}{27}a^3)}}}=$$

$$\sqrt[3]{288+\sqrt{(288^2-32^3)}}-\frac{-32}{\sqrt[3]{288+\sqrt{(288^2-32^3)}}}=$$

$12=p$; *and therefore* $y=16$, *or* $\sqrt{y}=4$, $f=\frac{-6}{2}+\frac{16}{2}+\frac{16}{8}=7$, *and* $g=\frac{-6}{2}+\frac{16}{2}-\frac{16}{8}=3$. *Whence* $e=4$, $f=7$, *and* $g=3$.

In the method of *Des Cartes*, above explained, all biquadratic equations are ſuppoſed to be generated from the multiplication of two quadratic ones: but according to the following way, which is taken from *Simpſon's Algebra*, every ſuch equation is conceived to ariſe by taking the difference of two complete ſquares.

Here, the general equation $x^4+ax^3+bx^2+cx+d=0$ being propoſed, we are to aſſume $(x^2+\frac{1}{2}ax+A)^2-(Bx+C)^2=x^4+ax^3+bx^2+cx+d$; in which A, B, and C, repreſent unknown quantities to be determined.

Then, $x^2+\frac{1}{2}a^2+A$, and $Bx+C$ being actually involved, will give
$$\left.\begin{array}{l} x^4+ax^3+2Ax^2 \\ +\frac{1}{4}a^2x^2+aAx+A^2 \\ -B^2x^2-2BCx-C^2 \end{array}\right\}=x^4+ax^3+bx^2$$
$+cx+d$: from whence, by equating the homologous terms, we ſhall have

1. $2A+\frac{1}{4}a^2-B^2=b$, or $2A+\frac{1}{4}a^2-b=B^2$
2. $aA-2BC=c$, or $aA-c=2BC$;
3. $A^2-C^2=d$, or $A^2-d=C^2$

Let now the firſt and laſt of theſe equations be multiplied together, and the product will, evidently, be equal to $\frac{1}{4}$ of the ſquare of the ſecond; that is $2A^3+(\frac{1}{4}a^2-b)\times A^2-2dA-(\frac{1}{4}a^2-b)\times d=(B^2C^2)=\frac{1}{4}\times(a^2A^2-2acA+c^2)$.

4. *To find the roots of the two quadratic equations* $x^2+ex+f=0$, *and* $x^2-ex+g=0$.

$$x^2+ex+f=x^2+4x+7=0$$
$$x^2-ex+g=x^2-4x+3=0$$

In the first of these $x=-2+\sqrt{-3}$, *or* $-2-\sqrt{-3}$.
And in the second $x=3$, *and* 1.

Therefore 3, 1, $-2+\sqrt{-3}$, *or* $-2-\sqrt{-3}$ *are the four roots of the equation* $x^4-6x^2-16x+21=0$.

And if unity be added to each of them, we shall have 4, 2, $-1+\sqrt{-3}$, *and* $-1-\sqrt{-3}$ *for the roots of* $z^4-4z^3-8z+32=0$, *the equation proposed; the two last of which are impossible.*

2. Given $x^4+2x^3-7x^2-8x+12=0$, to find the values of x. *Ans.* $x=1$, 2, -3 *and* -2.

Whence by denoting the given quantities $\frac{1}{4}ac-d$, and $\frac{1}{4}c^2+d\times(\frac{1}{4}a^2-b)$ by k and l, respectively, there will arise this cubic equation $A^3-\frac{1}{2}bA^2+kA-\frac{1}{8}l=0$; by means of which the value of A may be determined; and therefore, from the preceding equations, both B and C will also be known; B being found from thence $=\sqrt{(2A+\frac{1}{4}a^2-b,)}$ and $C=\frac{aA-c}{2B}$.

The several values of A, B, and C being thus found, that of x will also be readily obtained; for $(x^2+\frac{1}{2}ax+A)^2-(Bx+C)^2$ being universally in all circumstances of x, equal to $x^4+ax^3+bx^2+cx+d$, it is evident, that, when the value of x is taken such that the latter of these expressions becomes equal to nothing, the former must likewise be equal to nothing; and consequently $(x^2+\frac{1}{2}ax+A)^2=(Bx+C)^2$.

And therefore, by extracting the square root of both sides of the equation, we shall have $x^2+\frac{1}{2}ax+A=\pm Bx\pm C$; or $x=\pm\frac{1}{2}B-\frac{1}{4}a\pm[(\frac{1}{4}a\mp\frac{1}{2}B)^2\pm C-A]^{\frac{1}{2}}=\pm\frac{1}{2}B-\frac{1}{4}a\pm(\frac{1}{16}a^2\pm\frac{1}{4}aB+\frac{1}{4}B^2\pm C-A)^{\frac{1}{2}}$, which exhibits all the different roots of the given equation, according to the variation of the signs.

3. Given $x^4-25x^2+60x-36=0$, to find the values of x. *Ans.* 1, 2, 3, *and* -6.

4. Given $y^4-8y^3+14y^2+4y-8=0$, to find the values of y.

Ans. $y=3+\sqrt{5}$, $3-\sqrt{5}$, $1+\sqrt{3}$, *and* $1-\sqrt{3}$.

5. Given $x^4+12x-17=0$, to find the values of x.
Ans. $x=\frac{1}{2}\sqrt{2}\pm\sqrt{(-3\sqrt{2}-\frac{1}{2})}$, *and also* $-\frac{1}{2}\sqrt{2}\pm\sqrt{(3\sqrt{2}-\frac{1}{2})}$.

6. Given $x^4-4x^3-3x^2-4x+1=0$, to find the values of x. *Ans.* $x=\frac{-1\pm\sqrt{-3}}{2}$ *and* $\frac{5\pm\sqrt{21}}{2}$.

7. Given $x^4-10x^3+35x^2-50x+24=0$, to find the values of x. *Ans.* 1, 2, 3, *and* 4.

This method will be found to have many advantages over that given above. In the first place, there is no necessity for being at the trouble of exterminating the second term of the equation, in order to prepare it for a solution; secondly, the equation $A^3-\frac{1}{2}bA^2+kA-\frac{1}{2}l=o$, here brought out, is of a more simple form than that derived from the former method: and thirdly, the value of A in this equation will always be *commensurate* and *rational;* not only when all the roots of the given equation are *commensurate*, but also when they are *irrational*, and even *impossible.*

EXAMPLE. *Let there be given* $x^4+12x-17=0$, *to find the value of* x.

Here, by comparing this with the general equation $x^4+ax^3+bx^2+cx+d=o$, we shall have $a=o$, $b=o$, $c=12$, and $d=17$; and therefore $k=\frac{1}{4}ac-d=17$, $l=\frac{1}{4}c^2+d\times(\frac{1}{4}a^2-b)=36$, and $A^3-\frac{1}{2}bA^2+kA-\frac{1}{2}l=A^3+17A-18=o$.

And, from this equation, A will be found equal to 1; and therefore $B=(2A+\frac{1}{4}a^2-b)^{\frac{1}{2}}=\sqrt{2}$, $C=\frac{aA-c}{2B}=\frac{-12}{2\sqrt{2}}=-3\sqrt{2}$, and $x=\pm\frac{1}{2}\sqrt{2}\pm(\frac{1}{2}\mp3\sqrt{2}-1)^{\frac{1}{2}}=\pm\frac{1}{2}\sqrt{2}\mp(\pm3\sqrt{2}-\frac{1}{2})^{\frac{1}{2}}$.

8. Given $x^4-6x^3-58x^2-114x-11=0$, to find th values of x.

Ans. $x=\pm\frac{5}{2}\sqrt{3}+\frac{3}{2}\pm\sqrt{(17\pm\frac{21}{2}\sqrt{3})}$.

9. Given $x^4-3x^2-4x-3=0$, to find the value of x.

Ans. $x=\frac{1\pm\sqrt{3}}{2}$ *and* $\frac{-1\pm\sqrt{-3}}{2}$.

PROBLEM VII.

To find the roots of equations in general, by the method of approximation and converging series.

RULE.*

1. Find, by trial, a number nearly equal to the root required.

In some particular cases of this rule, the roots may be found by means of quadratics only.

Several other methods of solving biquadratic equations have been invented by different authors; one of the most ingenious of which is that given by *M. Euler*, in page 664 of his *Elemens de Algebre.*

Equations of five or more dimensions may sometimes be reduced to those of an inferior degree; but the process will be exceedingly tedious, as no general rule can be given for resolving them.

* The rules hitherto given for finding the roots of equations, are either very troublesome and laborious, or else confined to particular cases; but this method, by converging series, is universal, extending to all kinds of equations whatever; and, though not accurately true, gives the value sought to any assigned degree of exactness.

The method of obtaining the roots of equations, by approximation, was first made use of by *Vieta.*

2. Call the number, thus found, r, and let z be put equal to the difference between r and the true root x.

3. Instead of x, in the given equation, substitute its equal $r \pm z$, and there will arise a new equation, affected only with z and known quantities.

4. Reject all those quantities in which there are two or more dimensions of z, and the value of z will be found by means of a simple equation.

5. Add the value of z, thus found, to the value of r, and it will give the root required *nearly*.

6. If this root is not sufficiently near the truth, repeat the operation, by substituting it instead of r, in the equation exhibiting the value of z, and it will give a *second correction* for the root required.

EXAMPLES.

1. Given $x^2-5x-31=0$, to find the value of x by approximation.

The root, found by trial, is nearly equal to 8;

Let, therefore, $8=r$, *and* $r+z=x$; *then*

$$x^2=r^2+2rz+z^2$$
$$-5x=-5r-5z$$
$$-31=-31$$

$$r^2+2rz-5r-5z-31=0;$$

or $z=\frac{31+5r-r^2}{2r-5}=\frac{31+40-64}{16-5}=\frac{7}{11}=.6$, and $x=8.6$ *nearly*.

And, again, if 8.6 *be substituted in the place of* r *in the last equation, we shall have*

$z=\frac{31+5r-r^2}{2r-5}=\frac{31+43-73.96}{17.2-5}=\frac{.04}{12.2}=.0032$, *and* $x=8.6+.0032=8.6032$ *nearly*.

And, if this value be again substituted for r, it will give $z = .0000077808$, *and* $x = 8.603277808$; *and so on to any degree of exactness.*

2. Given $x^3 + x^2 + x = 90$, to find the value of x by approximation.

The root, found by trial, is nearly equal to 4;
Let, therefore, $4 = r$, *and* $r + z = x$, *then,*

$$x^3 = r^3 + 3r^2z + 3rz^2 + z^3$$
$$x^2 = r^2 + 2rz + z^2$$
$$x = r + z$$

$$r^3 + 3r^2z + r^2 + 2rz + r + z = 90;$$

$$or,\ z = \frac{90 - r^3 - r^2 - r}{3r^2 + 2r + 1} = \frac{90 - 64 - 16 - 4}{48 + 8 + 1} = \frac{6}{57} = .10,$$

and $x = 4.1$ *nearly.*

And again, if 4.1 *be substituted in the place of r, in the last equation, we shall have.*

$$\frac{90 - r^3 - r^2 - r}{3r^2 + 2r + 1} = \frac{90 - 68.921 - 16.81 - 4.1}{50.43 + 8.2 + 1.} = .00283,$$

and $x = 4.1 + .00283 = 4.10283$ *nearly; and so on to any degree of exactness required.*

3. Given $x^2 + 20x = 100$, to find the value of x by approximation *Ans.* $x = 4,1421356$.

4. Given $x^3 + 10x^2 + 5x = 2600$, to find the value of x by approximation. *Ans.* 11,00673.

5. Given $x^3 + 2x^2 - 23x - 70 = 0$, to find the value of x. *Ans.* $x = 5.1349$.

6. Given $x^3 - 15x^2 + 63x - 50 = 0$, to find the value of x. *Ans.* $x = 1.028039$.

7. Given $x^4 - 3x^2 - 75x = 10000$, to find the value of x. *Ans.* $x = 10.2615$.

8. Given $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$, to find the value of x. *Ans.* $x = 8.4144$.

RULE II.

1. Affume the general equation $az+bz^2+cz^3+dz^4$, &c. $=p$; where z is the converging quantity, and a, b, c, d, &c. co-efficients whofe values are known.

2. Then will $\frac{ap}{a^2+bp}$ be an approximation of the firft degree.

3. And, if s be put $=\frac{b}{a}-\frac{c}{b}$ we fhall have $\frac{(a+sp)\times p}{a^2+(b+as)\times p}$ for an approximation of the fecond degree.

4. And, in like manner, if w put $=\frac{2b}{a}+\frac{ad-bc}{b^2-ac}$, then will $\frac{ap\times(a+wp)}{a\times(a^2+(b+aw)\times p)+(w-s)\times p^2}$ be an approximation of the third degree, &c.

EXAMPLES.

1. Given $x^2+20x=100$, to find the value of x.

The root, found by trial, is nearly equal to 4;

Let therefore $4+z=x$, *and, by fubftitution, the equation will become* $28z+z^2=4$.

Whence, from the rule $a=28$, $b=1$, $c=0$, *&c. and* $p=4$.

Therefore $z=\frac{ap}{a^2+bp}=\frac{112}{788}=\frac{28}{197}=.14213$ *for the firft approximation.*

And since $s=\frac{b}{a}-\frac{c}{b}=\frac{1}{28}$, *we shall have* $z=$

$$\frac{(a+sp)\times p}{a^2+(b+as)\times p}=\frac{(28+\frac{1}{7})\times 4}{28\times 28+(1+1)\times 4}=\frac{28+\frac{1}{7}}{28\times 7+2}$$

$=\frac{197}{1386}=.1421356\frac{4}{}$ *for the second approximation.*

And, in like manner, since $w=\frac{2b}{a}+\frac{ad-bc}{b-ac}=\frac{1}{14}$, *it*

will be $z=\frac{ap\times(a+wp)}{a\times(a^2+(b+aw)\times p+(w-s)\times p^2}=$

$$\frac{28\times 4\times(28+\frac{2}{7})}{28\times(784+12)+\frac{4}{7}}=\frac{28\times(28+\frac{1}{7})}{7\times 796+\frac{1}{7}}=\frac{28\times 198}{49\times 796+1}=$$

$\frac{5544}{39005}=.1421356236$; *or* $x=4.1421356236$ *for the root required, extremely near.*

2. Given $x^2+4x-10=100$, to find the value of x.
Ans. 8.677078.

3. Given $x^3-17x^2+54x=350$, to find the value of x.
Ans. 14.95407.

4. Given $x^3-2x-5=0$, to find the value of x.
Ans. 2.094551.

5. Given $2y^4-16y^3+40y^2-30y=-1$, to find the value of y.
Ans. $y=1.284724$.

6. Given $x^5+2x^4+3x^3+4x^2+5x=54321$, to find the value of x.
Ans. $x=8.414455$.

In the same manner the roots of other equations may be approximated; but to avoid trouble in preparing the equation for a solution, all such powers of the converging quantity z, as would rise higher than the degree, or order of the approximation you intend to work by, may be every where neglected.

PROBLEM VIII.

To extract the root of any pure power in numbers.

RULE.*

1. Let $m=$ the number whofe root is required; $r=$ neareft root which can be found by trial; and $n=$ to the index.

2. Then, by putting $v=\frac{nr^n}{m-r^n}$, we fhall have $x=r+\frac{r\times(6v+n+1)}{v\times(6v+4n-2)}=$ root, *nearly*; or $x=r+\frac{r\times(v+n)}{v\times(2v+2n-1)+\frac{1}{6}\times(n-1)\times(2n-1)}$ *extremely near.*

EXAMPLES.

1. Given $x^2=2$; or, which is the fame thing, let the fquare root of 2 be found.

Suppofe the root found by trial, to be 1.4; *then we fhall have* $m=2$; $r=1.4$, $n=2$, *and* $v=\frac{2\times1.96}{2-1.96}=98$.

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* One of the moft convenient rules for practice, which has yet been difcovered, is the following: $(n+1)r^n+(n-1)N : (n+1)N+(n-1)r^n :: r$: the true root, nearly.

Where it may be obferved that $N=$ given number; $r=$ neareft root, found by trial; and $n=$ index, as before.

And, therefore, $x = r + \frac{r \times (6v + n + 1)}{v \times (6v + 4n - 2)} = 1.4 + \frac{1.4 \times 591}{98 \times 594} = 1.4 + \frac{197}{70 \times 198} = 1.4 + \frac{197}{13860} = 1.41421356$ *= root required nearly.*

And if the second approximation be used, the root will be found $= 1.41421356236$, *which is true to the last place of decimals.*

2. Given $x^3 = 500$; or let it be required to extract the cube root of 500.

Suppose the root, found by trial, to be 8; *then we shall have* $m = 500$, $r = 8$, $n = 3$, *and* $v = \frac{3 \times 512}{-12} = -128$;

And, therefore, $x = r + \frac{r \times (6v + n + 1)}{v \times (6v + 4n) - 2} = 8 - .063 = 7.93$ *for the first approximation.*

Or $x = r + \frac{r \times (2v + n)}{(2v + 2r - 1) \times v + \frac{1}{6} \times (n - 1) \times (2n - 1)} = 8 - \frac{6072}{96389} = 7.937005259936$ *for the second approximation which is true to the last place of decimals.*

3. Let it be required to find the cube root of 2.

Ans. 1.259921.

4. Required the cube root of 117? *Ans.* 4.89097.

5. What is the surfolid, or 5th root, of 125000.

Ans. 10.456389.

6. It is required to find the 7th root of 100000.

Ans. 5.179474679.

7. It is required to find the 365th root of 1.05.

Ans. 1.00013366.

PROBLEM IX.

To find the root of an exponential equation.

RULE.*

1. Find, by trial, two numbers, as near the true root as possible, and substitute them in the given equation instead of the unknown quantity, marking the errors which arise from each of them.

2. Multiply the difference of the two numbers, found by trial, by the least error, and divide the product by the difference of the errors, when they are alike, and by their sum when they are unlike.

3. Add the quotient, last found, to the number belonging to the least error, when that number is too little, and subtract it when too great, and the result will give the true root *nearly*.

4. Take this root and the nearest of the former, and, by proceeding in like manner, a root will be had still nearer than before; and so on to any degree of exactness required.

EXAMPLES.

1. Given $x^x=100$ to find the value of x by approximation.

By the nature of logarithms $x\times$ *log.* $x=$ *log.* $100=2$.

And, since x *is found by trial to be greater than* 3 *and less than* 4.

Let, therefore, 3.5 *and* 3.6 *be the two supposed values of* x.

* The rule for solving exponential equations was invented by *M. John Bernoulli*, and published in the *Leipsic* Acts, 1697.

Then the log. $x=$ *log.* $3.5=.5440680$; *and* $x \times$ *log.* $x=1.9042380$

$$2$$

$$-.0957620=\text{ 1st error too little}$$

And the log. $x=$ *log.* $3.6=.5563025$; *and* $x \times$ *log.* $x=2.0026890$

$$2$$

$$.0026890=\text{ 2d. error, too great.}$$

1st number	3.5	*1st error*	$-.095762$
2d number	3.6	*2d error*	$+.002689$
	$0.1=$ *diff.*		$.098451=$ *sum.*

$$\frac{0.1 \times .002689}{.098451}=00273=\text{ correction.}$$

2d number 3.60000
correction $-.00273$

$$3.59727=x=\text{ root nearly.}$$

Again, suppose $x=3.597$; *then we shall have* *log.* $x=.5559404$, *and* $x \times$ *log.* $x=1.9997176$, *which subtracted from* 2 *gives* .0002824, *the third error, too little.*

2d number	3.600	*2d error*	$+.0026890$
3d number	3.597	*3d error*	-0002824
	$.003=$ *diff.*		$.0029714=$ *sum,*

$$\frac{.003 \times .0002824}{.0029714}=.000285\text{ the correction.}$$

3d number 3.597000
correction $+0.000285$

$$3.597285=x=\text{ root required nearly.}$$

2. Given $x^x=123456789$ to determine the value of x.
Ans. $x=8.6400268$

OF

INDETERMINATE OR UNLIMITED PROBLEMS.

A problem is ſaid to be *indeterminate* or *unlimited*, when the equations, expreſſing the conditions of a queſtion, are leſs in number than the unknown quantities to be determined.

And though ſuch kind of problems are capable of innumerable anſwers, yet the reſults, in whole numbers, are generally limited to ſome determinate number, and may be obtained as follows.

PROBLEM I.

To find the values of x and y, in the equation $ax=by+c$; where a, b, and c, are given numbers, which admit of no common diviſor.

RULE.*

1. Let w ſtand for a whole number, and reduce the equation to $x=\frac{by+c}{a}=wh$

2. Make $\frac{by+c}{a}=\frac{dy+f}{a}$, by throwing all whole numbers out of it, till d and f be each leſs than a.

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* This rule is founded on the following obvious principles:

That the ſum, difference, or product, of any two whole numbers, is a whole number.

And that if a number meaſures the whole of any number, and a part of it, it will alſo meaſure the remaining part.

3. Subtract $\frac{dy+f}{a}$, or some multiple of it, from $\frac{ay}{a}$, $\frac{2ay}{a}$, $\frac{3ay}{a}$, or any other multiple of y that comes near the former, and the remainder will be a whole number.

4. Take this remainder, or any multiple of it, from some of the foregoing fractions, or from any whole number, which is nearly equal to it, and the remainder, in this case, will also be a whole number.

5. Proceed in the same manner, with this last remainder; and so on till the co-efficient of y becomes equal to 1; or $\frac{y+g}{a}=wh.=p$.

6. Then will $y=ap-g$; where p may be any whole number whatever; and as the value of y is now known, that of x may also be found from the given equation.

EXAMPLES.

1. Given $19x=14y-11$, to find x and y in whole numbers.

First, $x=\frac{14y-11}{19}=wh.$; *and* $\frac{19y}{19}=wh.$

Then by subtraction, $\frac{19y}{19}-\frac{14y-11}{12}=\frac{5y+11}{19}=wh.$

Again, $\frac{5y+11}{19}\times4=\frac{20y+44}{19}=\frac{20y+6}{19}+2=wh.$

and, by rejecting the 2, $\frac{20y+6}{19}=wh$

Therefore $\frac{20y+6}{19}-\frac{19y}{19}=\frac{y+6}{19}=wh.=p.$

And $y=19p-6$; *where, if* p *be taken* $=1$, *for the least affirmative value of* y, *we shall have* $y=13$, *and* $x=9$, *the answer.*

2. Given $3x=8y-16$, to find the values of x and y in whole numbers.

Here, $x=\frac{8y-16}{3}=2y-5+\frac{2y-1}{3}=$ *wh. or* $\frac{2y-1}{3}$ $=$*wh.*

And $\frac{2y-1}{3}\times 2=\frac{4y-2}{3}=$*wh.* *But,* $\frac{3y}{3}$ *is also* $=$*wh.*

Therefore $\frac{4y-2}{3}-\frac{3y}{3}=\frac{y-2}{3}=$*wh.*$=p$,

Whence $y=3p+2$; *and by taking* $p=1$, *we shall have* $y=5$ *and* $x=8$, *the answer.*

3. Given $9x+13y=2000$, to find all the possible values of x and y in whole numbers.

First, $x=\frac{2000-13y}{9}=222-y+\frac{2-4y}{9}=$*wh.*

Or, by rejecting $222-y$, $\frac{2-4y}{9}=$ *wh. numb.*

And, $\frac{2-4y}{8}\times 2=\frac{4-8y}{9}=$*wh.* *But,* $\frac{9y}{9}$ *is also*$=$*wh.*

Therefore, $\frac{9y}{9}+\frac{4-8y}{9}=\frac{y+4}{9}=$*wh.* $=p$.

Whence $y=9p-4$; *and by taking* $p=1$, *we shall have* $y=5$ *and* $x=215$.

And, by adding 9 *continually to the last value of* y, *and subtracting* 13 *from that of* x, *all the possible answers will stand as follows:*

$$x=\begin{cases}215.189.163.137.111.85.59.33\\202.176.150.124.98.72.46.20\end{cases}.7$$

$$y=\begin{cases}5.23.41.59.77.95.113.131\\13.32.50.68.86.104.122.140\end{cases}.149$$

4. Given $24x=13y+16$, to find x and y in whole numbers. *Ans.* $x=5$, *and* $y=8$.

5. Given $14x=5y+7$, to find x and y in whole numbers. *Ans.* $x=3$, *and* $y=7$.

6. Given $27x=1600-16y$, to find x and y in whole numbers. *Ans.* $y=19$, *and* $x=48$.

7. Given $87x+256y=15410$, to find the leaſt value of x, and the greateſt of y, in whole poſitive numbers. *Ans.* $x=30$, *and* $y=$ 50.

8. Given $5x+7y+11z=224$, to find all the poſſible values of x, y, and z, in whole numbers.

Ans. The number of anſwers is 60.

9. To determine whether it be poſſible to pay 100l. in guineas and moidores only.

Ans. The queſtion is impoſſible.

10. How many different ways is it poſſible to pay 100l. in guineas and piſtoles only; a guinea being equal to 21s. and a piſtole to 17s. *Ans.* 6 *different ways.*

11. Forty-one perſons, men, women, and children, ſpent among them 40s. of which each man paid 4s. each woman 3s. and each child 4d. how many were there of each? *Ans.* 5 *men,* 3 *women, and* 33 *children.*

12. I owe my friend a ſhilling, and have nothing about me but guineas, and he has nothing but louis d'ors; the queſtion is, how muſt I acquit myſelf of the debt? the louis d'ors being valued at 17s. *Anſ. I muſt pay him* 13 *guineas, and he muſt give me* 16 *louis d'ors.*

13. To find in what year of Chriſt the cycle of the ſun was 8, the cycle of the moon 10, and the cycle of indiction 10. *Ans. In the year* 1767.

14. It is required to diſcharge a debt of 2651. with guineas and moidores only, ſo that there may be the leaſt number of pieces of each ſort; and to find what the whole will amount to, when paid every way it will poſſibly admit of. *Anſ. The number of ways* 36 *and the whole amount* 12636*l*

15. A vintner has wine at 2s. 1s. 10d. and 1s. 6d. *per* gallon: how much of each sort must he take, so as to make a mixture of 30 gallons, to be sold at 1s. 8d. *per* gallon? *Ans.* 16, 2, 12, 17, 4. 9, 18, 6, 6; *or* 19, 8, 3, *of each sort.*

16. To determine how many ways it is possible to pay 100l. in crowns, guineas, and moidores only.

Ans. 70734 *different ways.*

PROBLEM II.

To find such a whole number x, as being divided by the given numbers, a, b, c, &c. shall leave the given remainders f, g, h, &c.

RULE.

1. Subtract each of the remainders from x, and divide the difference by a, and there will result $\frac{x-f}{a}$, $\frac{x-g}{a}$, $\frac{x-h}{a}$, &c. = *whole numbers.*

2. Call the value of x, in the first fraction, p, and substitute this quantity in the place of x, in the second fraction.

3. Find the least value of p, in the second fraction, by the last problem, and call it r.

4. Let the value of x be found in terms of r, and substitute this quantity in the place of x in the third fraction.

5. Find the least value of r in the third fraction, and call it s; and the value of x in terms of s, being substituted for x in the fourth fraction, and so on, will give the whole number required.

EXAMPLES.

1. To find the leaſt whole number, which being divided by 17, ſhall leave a remainder of 7; but, being divided by 26, the remainder ſhall be 13.

Let $x=$ *number required.*

Then $\frac{x-7}{17}$, *and* $\frac{x-13}{26}=$ *whole numbers.*

And, by putting $\frac{x-7}{17}=p$, *we ſhall have* $x=17p+7$;

Which value of x *being ſubſtituted in the 2d fraction,*

gives $\frac{17p-6}{26}=wh.$ *But* $\frac{26p}{26}$ *is alſo* $=wh.$

And, therefore, $\frac{26p}{26}-\frac{17p-6}{26}=\frac{9p+6}{26}=wh.$

Or $\frac{9p+6}{26}\times 3=\frac{27p+18}{26}=p+\frac{p+18}{26}=$ *wh. number.*

And, by rejecting p, *we have* $\frac{p+18}{26}=wh.=r.$

Hence $p=26r-18$, *and by taking* $r=1$, *we ſhall have* $p=8$.

And, conſequently $x=17\times 8+7=143$, *the number required.*

2. To find a number which being divided by 11, 19, and 29, the remainders ſhall be 3, 5, 10.

Let $x=$ *number required.*

Then $\frac{x-3}{11}$, $\frac{x-5}{19}$ *and* $\frac{x-10}{29}=$ *whole numbers.*

And, by putting $\frac{x-3}{11}=p$, *we ſhall have* $x=11p+3$;

Which value of x, being substituted in the 2d fraction, *gives* $\frac{11p-2}{19}$=*wh. or* $\frac{11p-2}{19}\times 2=\frac{22p-4}{19}=p+\frac{3p-4}{19}$=*wh. And, by rejecting p, we shall have* $\frac{3p-4}{19}$=*wh. Also* $\frac{3p-4}{19}\times 6=\frac{18p-24}{19}=\frac{18p-5}{19}-1$=*wh. or by rejecting the* 1, $\frac{18p-5}{19}$=*wh. number.*

But, $\frac{19p}{19}$ *is likewise,* =*wh. and therefore,* $\frac{19p}{19}-\frac{18p-5}{19}=\frac{p+5}{19}$=*wh. number, which let be put* $=r$.

Then, $p=19r-5$, *and* $x=(19r-5)\times 11+3=209r-52$.

And by substituting this value of x in the 3d fraction, we shall have $\frac{209r-52}{29}=7r-2+\frac{6r-4}{29}$=*wh. which, by neglecting* $7r-2$, *gives* $\frac{6r-4}{29}$=*wh. number.*

But, $\frac{6r-4}{29}\times 5=\frac{30r-20}{29}=r+\frac{r-20}{29}$=*wh. or by rejecting* r, $\frac{r-20}{29}$=*wh. which let be put* $=s$.

Then $r=29s+29$; *and by putting* $s=0$, *we shall have* $r=20$.

And, consequently, $x=209\times 20-52=4128$;

Therefore, 4128= *number required.*

3. To find the least whole number, which, being divided by 19, shall leave a remainder of 7; but, being divided by 28, the remainder shall be 13. *Ans.* 349.

4. To find a number which being divided by 3, 5, 7, and 2, will leave the remainders 2, 4, 6, and 0, respectively. *Ans*. 104.

5. To find the least whole number, which, being divided by 16, 17, 18, 19, and 20, shall leave 6, 7, 8, 9, and 10 remainders. *Ans*. 232550.

6. To find the least whole number, which, being divided by the nine digits respectively, shall leave no remainders. *Ans*. 2520.

7. To find the least whole number, which being divided by 2, 3, 5, 7, and 11, shall leave 1, 2, 3, 4, and 5, for remainders. *Ans*. 1523.

DIOPHANTINE PROBLEMS.

* *Diophantine problems* are those which relate to the finding of square and cube numbers, &c. and are such as are generally capable of a great variety of answers. They are so called from their inventor *Diophantus* of *Alexandria* in *Egypt*, who flourished in or about the third century, and is the first writer on Algebra we meet with amongst the ancients.

These questions are so exceedingly curious and abstruse, that nothing less than the most refined Algebra, applied

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* That *Diophantus* was not the inventor of algebra, as has been generally imagined, is obvious; for his method of applying it is such, as could only have been used in a very advanced state of the science.

He no where speaks of the fundamental rules and principles, as an inventor certainly would have done, but treats of it as an art already sufficiently known; and seems to intend, not so much to teach it, as to cultivate and improve it, by solving such questions as, before his time, had been thought too difficult to be surmounted.

with the utmoſt ſkill and judgment, can ſurmount the difficulties which attend them. And, in this way, no man has ever extended the limits of the analytic art further than *Diophantus*, or diſcovered greater penetration or judgment in the application of it.

When we conſider his work with attention, we are at a loſs which to admire moſt, his wonderful ſagacity and peculiar artifice, in forming ſuch poſitions as the nature of the problems required, or the more than ordinary ſubtilty of his reaſoning upon them.

Every particular queſtion puts us upon a new way of thinking, and furniſhes a freſh vein of analytical treaſure, which cannot but be very inſtructive to the mind, in conducting it through almoſt all difficulties of this kind, whenever they occur.

The following method of reſolving theſe queſtions will be found of conſiderable ſervice; but no general rule can be given, that will ſuit all caſes; and therefore the ſolution muſt often be left to the ſagacity and ſkill of the learner.

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* It is probable, therefore, that Algebra was known in the world long before the time of *Diophantus*; but that the works of preceding writers have been deſtroyed by the ravages of time, or the depredations of ignorant barbarians.

His *Arithmetics*, out of which theſe problems were moſtly collected, conſiſted originally of thirteen books; but the firſt ſix only are now extant. The beſt edition is ſaid to be that publiſhed at *Paris*, by Monſieur *Bachet*, in the year 1621. In this work, the ſubject is ſo ſkilfully handled, that the moderns, notwithſtanding their other improvements, have been able to do little more than explain and illuſtrate his method.

Thoſe who have ſucceeded beſt, in this reſpect, are *Vieta*, *Brauncker*, *Kerſey*, *De Billy*, *Ozanam*, *Preſtet*, *Saunderſon*, *Fermat*, and *Euler*. The laſt of whom, in particular, has amplified and illuſtrated the Diophantine Algebra in as clear and ſatisfactory a manner as the ſubject ſeems to admit.

RULE.

1. For the root of the ſquare or cube required, put one or more letters ſuch, that when they are involved, either the given number, or the higheſt power of the unknown quantity, may vaniſh from the equation; and then, if the unknown quantity be but of one dimenſion, the problem will be ſolved by reducing the equation.

2. But if the unknown quantity be ſtill a ſquare, or a higher power, ſome other new letters muſt be aſſumed to denote the root; with which proceed as before; and ſo on till the unknown quantity is but of one dimenſion; and from this all the reſt will be determined.

EXAMPLES.

1. * To divide a given ſquare number (100) into two ſuch parts, that each of them may be a ſquare number.

Let $x^2(=\square)$ be one of the parts, and then $100-x^2$ will be the other part, which is alſo to be a ſquare number.

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* If $x-10$ had been made the ſide of the ſecond ſquare, in this queſtion, inſtead of $2x-10$, the equation would have been $x^2-20x+100=100-x^2$; in which caſe, x, the ſide of the firſt ſquare would have been found $=10$, and $x-10$, or the ſide of the ſecond ſquare $=0$; and for this reaſon the ſubſtitution, $x-10$, was avoided; but $3x-10$, $4x-10$, or any other quantity of the ſame kind, would have ſucceeded equally as well as the former; though, in ſome caſes, the reſults would have been leſs ſimple.

Assume the side of this second square $=2x-10$.
Then will $100-x^2=(2x-10)^2=4x^2-40x+100$;
And, by reduction, $x=8$, *and* $2x-10=6$.
Therefore 64 *and* 36 *are the parts required.*

THE SAME GENERALLY.

Let a^2 = *given square number,* x^2 $(=\square)$ = *one of its parts, and* a^2-x^2 = *the other, which is also to be a square number.*

Assume the side of this second square $=rx-a$,
then will $a^2-x^2=(rx-a)^2=r^2x^2-2arx+a^2$;

And, by reduction, $x=\dfrac{2ar}{r^2+1}$, *and* $rx-a=\dfrac{2ar^2}{r^2+1}-a$

$=\dfrac{2ar^2}{r^2+1}-\dfrac{ar^2+a}{r^2+1}=\dfrac{ar^2-a}{r^2+1}=$ *to a square number.*

Therefore $\left(\dfrac{2ar}{r^2+1}\right)^2$ *and* $\left(\dfrac{ar^2-a}{r^2+1}\right)^2$ *are the parts required;*

where a and r may be any numbers, taken at pleasure.

2. * To divide a given number (13) consisting of two known square numbers (9 and 4) into two other square numbers.

If s and r be any two unequal numbers, of which s is the greater then will $2rs$, s^2-r^2 and s^2+r^2 be the perpendicular, base, and hypothenuse of a right-angled triangle.

And from this canon two square numbers may be found, whose sum or difference shall be square numbers; for $(2rs)^2+(s^2-r^2)^2=(s^2+r^2)^2$, and $(r^2+s^2)^2-(2rs)^2=(s^2-r^2)^2$, or $(s^2+r^2)^2-(s^2-r^2)^2=(2rs)^2$; and this when s and r are any numbers whatever.

* This question is considered by *Diophantus* as a very important one, being made the foundation of most of his other problems.—In

For the side of the first square sought, put $rx-3$; and for the side of the second, $sx-2$; r being the greater number, and s the less.

Then will $(rx-3)^2+(sx-2)^2=(r^2x^2-6rx+9)+(s^2x^2-4sx+4)=(r^2+s^2)x^2-(6r+4s)x+13=13$, *or* $(r^2+s^2)x^2=(6r+4s)x$.

And from this equation, by reduction, x is found $=\dfrac{6r+4s}{r^2+s^2}$.

Whence $rx-3=\dfrac{6r^2+4rs}{r^2+s^2}-3=\dfrac{3r^2+4rs-3s^2}{r^2+s^2}=$ *side of the first square sought.*

And, $sx-2=\dfrac{6rs+4s^2}{r^2+s^2}-2=\dfrac{6rs-2r^2+2s^2}{r^2+s^2}=$ *side of the second.*

So that if r be taken $=2$, and $s=1$, we shall have $\dfrac{3r^2+4rs-3s^2}{r^2+s^2}=\dfrac{17}{5}$, *and* $\dfrac{6rs-2r^2+2s^2}{r^2+s^3}=\dfrac{6}{5}$ *for the sides of the squares, in numbers, as was required.*

If a^2+b^2 be put equal to the number to be divided, the general solution may be given in exactly the same manner.

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the solution of it, given above, the values of r and s may be taken equal to any numbers whatever, provided the proportion of those numbers be not the same as that of $3(a)$ to $2(b)$, or $3+2(a+b)$ to $3-2(a-b)$. And the reason of this restriction is, that if r and s were so taken, the sides of the squares sought would come out the same as the sides of the known squares which compose the given number, and therefore the operation would be useless.

The excellent old *Kersey*, after amplifying and illustrating this problem in a variety of ways, concludes his chapter thus: "For a further account of this *rare* speculation, see *Andersonus*, Theorem 2 of *Vieta's mysterious* doctrine of Angular Sections; and likewise, *Herigonius*, at the latter end of the first *tome* of his *Cursus Mathematicus*.

3. To find two square numbers, whose difference shall be equal to any given number (d).

Let d be resolved into any two unequal factors a and b; a being the greater and b the less.

Also put x for the side of the less square sought, and $x+b=$ side of the greater.

Then $(x+b)^2-x^2=x^2+2bx+b^2-x^2=2bx+b^2=d$ (ab).

And if this be divided by b, we shall have $2x+b=a$.

Whence $x=\frac{a-b}{2}=$ *side of the less square sought,*

$$\text{and } x+b=\frac{a-b}{2}+b=\frac{a+b}{2}=\text{ side of the greater.}$$

So that by putting $d=60$, and $a\times b=2\times30$, we shall have $\frac{30-2}{2}=14$, *and* $\frac{30+2}{2}=16$, *or* $(14)^2=196$, *and* $(16)^2=256$ *for the squares in numbers; and so far any difference or factors whatever.*

4. To find two numbers such, that, if either of them be added to the square of the other, the sum shall be a square number.

Let the numbers sought be x and y.

Then $x^2+y=\square$, *and* $y^2+x=\square$.

And, if, $r-x$ be assumed for the side for the first square x^2+y, we shall have $x^2+y=r^2-2rx+x^2$, *or* $y=r^2-2rx$.

$$\text{Whence } 2rx=r^2-y \text{ or } x=\frac{r^2-y}{2r}.$$

Again if $y+s$ be assumed for the side of the second square, we shall have $y^2+\frac{r^2-y}{2r}(y^2+x)=(y+s)^2=y^2+2sy+s^2$

$$\text{Whence, } \frac{r^2-y}{2r}=2sy+s^2, \text{ or } r^2-y=4rsy+2rs^2.$$

And consequently, $y=\frac{r^2-2rs^2}{4rs+1}$ *and* $x=\frac{r^2-y}{2r}=\frac{2r^2s+s^2}{4rs+1}$.

So that $\frac{r^2-2rs^2}{4rs+1}$ *and* $\frac{2r^2s+s^2}{4rs+1}$ *are the numbers required; where r and s, may be taken at pleasure, provided r be greater than* $2s^2$.

5. To find two numbers, whose sum and difference shall be both square numbers.

Let x and x^2-x *be the two numbers sought.*

Then, since their sum is evidently a square number, one of the conditions of the question will be answered.

There remains, therefore, only their difference x^2-2x *to be made a square.*

And, if for the side of this square there be put $x-r$, *we shall have* $x^2-2rx+r^2=x^2-2x$, *or* $2rx-2x=r^2$.

Whence, $x=\frac{r^2}{2r-2}$ *and* $x^2-x=\left(\frac{r^2}{2r-2}\right)^2-\frac{r^2}{2r-2}$.

So that $\frac{r^2}{2r-2}$ *and* $\left(\frac{r^2}{2r-2}\right)^2-\frac{r^2}{2r-2}$ *are the numbers required; where r may be taken at pleasure, provided it be greater than 1.*

6. To find three numbers such, that not only the sum of all three of them, but also the sum of every two, shall be a square number.

Let $4x$, x^2-4x *and* $2x+1$, *be the three numbers sought.*

Then $(4x)+(x^2-4x)=x^2$, $(x^2-4x)+(2x+1)=x^2-2x+1$, *and* $(4x+x^2-4x+2x+1)=x^2+2x+1$, *are evidently squares.*

And, therefore, three of the conditions mentioned in the questions are accomplished.

Whence it remains only to make the quantity $(4x)+(2x+1)$, *or* $6x+1=$ *to a square.*

Let, therefore, $6x+1=a^2$; *and we shall have* $x=\frac{a^2-1}{6}$.

And, consequently, $\frac{4a^2-4}{6}\left(\frac{a^2-1}{6}\right)^2-\frac{4a^2-4}{6}$ *and* $\frac{2a^2-2}{6}+1$; *or* $\frac{2a-2}{3}$, $\frac{a^4-26a^2+25}{36}$, *and* $\frac{a^2+2}{3}$, *are the numbers required; where a may be taken at pleasure, provided it be greater than* 5.

7. To find three ſquare numbers ſuch, that the ſum of every two of them ſhall be a ſquare number.*

Let x^2, y^2, *and* z^2, *be the numbers sought;*

Then $x^2+z^2=\square$, $y^2+z^2, =\square$, *and* $x^2+y^2=\square$.

Or $\frac{x^2}{z^2}+1=\square$, $\frac{y^2}{z^2}+1=\square$, *and* $\frac{x^2}{z^2}+\frac{y^2}{z^2}=\square$.

And, by putting $\frac{x}{z}=\frac{s^2-1}{2s}$, *and* $\frac{y}{z}=\frac{r^2-1}{2r}$, *we shall have* $\frac{x^2}{z^2}+1=\frac{s^4+2s^2+1}{4s^2}$, *and* $\frac{y^2}{z^2}+1=\frac{r^4+2r^2+1}{4r^2}$, *which are both evidently squares; and therefore it remains only to make* $\frac{x^2}{z^2}+\frac{y^2}{z^2}=$ *square number.*

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* This queſtion is capable of a great variety of anſwers; but the leaſt roots which have yet been found, in whole numbers, are 44, 117, and 240 See *Elemens de Algebre*, par *M. Euler*, tome II. page 327; which is a work particularly calculated for the uſe of thoſe who wiſh to obtain a knowledge of Algebra without the aſſiſtance of a maſter.

$$But\ \frac{x^2}{z^2}+\frac{y^2}{z^2}=\left(\frac{s^2-1}{2s}\right)^2+\left(\frac{r^2-1}{2r}\right)^2=\frac{(s^2-1)^2}{4s^2}+\frac{(r^2-1)^2}{4r^2}=\frac{4r^2\times(s^2-1)^2+4s^2\times(r^2-1)^2}{4r^2s^2}=\text{ square No.}$$

Or $r^2\times(s^2-1)^2+s^2\times(r^2-1)^2=r^2\times(s+1)^2\times(s-1)^2+s^2\times(r+1)^2\times(r-1)^2=$ *to a square number.*

And by making $r-1=s+1$, *or* $r=s+2$, *we shall have* $(s+2)^2\times(s+1)^2\times(s-1)^2+s^2\times(s+3)^2\times(s+1)^2$ $=$ *to a square number.*

Or $(s+2)^2\times(s-1)^2+s^2\times(s+3)^2=2s^4+8s^3+6s^2-4s+4=$ *to a square number.*

Now, let the root of this square be assumed $=\frac{5}{4}s^2-s+2$,

Then, $2s^4+8s^3+6s^2-4s+4=(\frac{5}{4}s^2-s+2)^2=\frac{25}{16}s^4-\frac{5}{2}s^3+5s^2+s^2-4s+4$; *or* $2s^4+8s^3=\frac{25}{16}s^4-\frac{5}{2}s^3$; *or* $2s+8=\frac{25}{16}s-\frac{5}{2}$.

Whence $s=-24$, *and* $r=-22$.

$$And,\ \frac{x}{z}=\frac{s^2-1}{2s}=-\frac{575}{48},\ and\ \frac{y}{z}=\frac{r^2-1}{2r}=-\frac{483}{44};$$

$$Or\ x=-\frac{575z}{48},\ and\ y=-\frac{483z}{44}.$$

In order, therefore, to have the answer in whole numbers let $z=528$, *and we shall have* $x=6325$, *and* $y=5796$ *or* 528, 5796 *and* 6325, *for the roots of the squares required.*

8. To find a number x, such, that $x+1$ and $x-1$ shall be both square numbers. *Ans.* $x=\frac{5}{4}$.

9. To find a number x such, that $x+128$ and $x+192$ shall be both squares. *Ans.* $x=97$.

10. To find a number x such, that x^2+x and x^2-x may be both squares. *Ans.* $\frac{25}{24}$.

11. To find two numbers x and y ſuch, that $x+y$, x^2+y, and y^2+x may be all ſquares.

Ans. $x=\frac{1}{8}$ *and* $y=\frac{1}{16}$.

12. To find three ſquare numbers in arithmetical progreſſion. *Ans.* 1, 25, *and* 49.

13. To find three ſquare numbers in harmonical proportion. *Ans.* 1225, 49 *and* 25.

14. To find three numbers in arithmetical progreſſion ſuch, that the ſum of every two of them may be a ſquare number. *Ans.* $120\frac{1}{2}$, $840\frac{1}{2}$, *and* $1560\frac{1}{2}$.

15. To find three numbers ſuch, that, if to the ſquare of each of them the ſum of the other two be added, the three ſums ſhall be all ſquares. *Ans.* $\frac{8}{5}$, $\frac{16}{5}$ *and* 1.

16. To find two numbers in proportion as 8 is to 15, and ſuch that the ſum of their ſquares ſhall make a ſquare number. *Ans.* 576 *and* 1080.

17. To find four numbers ſuch, that, if a ſquare number (100) be added to the product of every two of them, the ſums ſhall be all ſquares.

Ans. 12, 32, 88, *and* 168.

18. To find two numbers ſuch, that their difference may be equal to the difference of their ſquares, and that the ſum of their ſquares ſhall be a ſquare number.

Ans. $\frac{4}{7}$ *and* $\frac{3}{7}$.

19. To find three numbers in geometrical proportion ſuch that every one of them being increaſed by a given number (19) ſhall make ſquare numbers.

Ans. 81, $\frac{5}{4}$ *and* $\frac{25}{1296}$.

20. To find two numbers ſuch that if their product be added to the ſum of their ſquares, it ſhall make a ſquare number. *Ans.* 5 *and* 3, 8 *and* 7, 16 *and* 5, &c.

21. To divide a given number (10) into four ſuch parts, that the ſum of every three of them may be a ſquare number. *Ans.* 1, 6, $\frac{186}{289}$ *and* $\frac{681}{289}$.

22. * To find three square numbers such that their sum being severally added to their three sides, shall make square numbers.

Ans. $\frac{44181}{62920}$, $\frac{13254}{62920}$, *and* $\frac{10881}{62920}$ = *roots required.*

23. To find two numbers such, that their sum being increased and lessened, either by their difference, or the difference of their squares, the sums and remainders shall be all squares. *Ans.* $\frac{49}{50}$, *and* $\frac{1}{50}$.

24. To find two numbers such, that not only each number, but also their sums and their difference, being increased by unity, shall be all square numbers.

Ans. 3024 *and* 5624.

25. To find three numbers such, that whether their sum be added to, or subtracted from, the square of each particular number, the numbers thence arising shall be all squares. *Ans.* $\frac{406}{96}$, $\frac{518}{96}$, *and* $\frac{701}{96}$.

26. To find three square numbers such, that the sum of their squares shall also be a square number.

Ans. 9, 16, *and* $\frac{144}{25}$.

27. To find three square numbers such that the difference of every two of them shall be a square number.

Ans. 485809, 34225, *and* 23409.

28. To divide any given cube number (8) into three other cube numbers. *Ans.* $\frac{64}{27}$, $\frac{125}{27}$, *and* 1.

29. Two cube numbers (8 and 1) being given, to find two other cube numbers, whose difference shall be equal to the sum of the given cubes.

Ans. $\frac{8000}{343}$, *and* $\frac{4913}{343}$.

30. To divide a given number (28) composed of two cube numbers (27 and 1) into two other cube numbers. *Ans.* $\frac{63284705}{21446828}$ *and* $\frac{28340511}{21446828}$ *the roots.*

31. To find three cube numbers such, that, if from every one of them a given number (1) be subtracted, the sum of the remainders shall be a square.

Ans. $\frac{4913}{3375}$, $\frac{21952}{3375}$ *and* 8.

* The answers to many of these questions cannot be given in whole numbers.

32. To find three numbers ſuch, that if they be ſeverally added to the cube of their ſum, the three ſums thence ariſing ſhall be all cubes.

Ans. $\frac{1538}{157464}$, $\frac{18577}{157464}$, *and* $\frac{23625}{157464}$.

33. To find three numbers in arithmetical proportion ſuch, that the ſum of their cubes ſhall be a cube.

Ans. 3, 4, 5, *or* 149, 256, 363, *&c.*

34. To find three cube numbers ſuch, that their ſum ſhall be a cube number.

Ans. 3^3, 4^3, *and* 5^3, *or* 21^3, 19^3, 18^3, *&c.*

35. To find two numbers ſuch, that their ſum ſhall be equal to the ſum of their cubes. *Ans.* $\frac{5}{7}$ *and* $\frac{8}{7}$.

OF THE

SUMMATION AND INTERPOLATION

OF

INFINITE SERIES.

The *doctrine of infinite series*, is a ſubject which has engaged the attention of the greateſt mathematicians in all ages; and is, perhaps, one of the moſt abſtruſe and difficult branches of abſtract mathematics.

To find the ſum of a ſeries, the number of whoſe terms is inexhauſtible, or infinite, has been conſidered by ſome as a paradox, or a thing impoſſible to be done. But this difficulty will be eaſily removed, by conſidering that every finite magnitude whatever is diviſible *in infinitum*, or conſiſts of an infinite number of parts whoſe aggregate, or ſum, is equal to the quantity firſt propoſed.

A number actually infinite is, indeed, a plain contradiction to all our ideas; for any number which we can poſſibly conceive, or of which we have any notion, muſt always be determinate and finite; ſo that a greater may be ſtill aſſigned, and a greater after this;

and ſo on, without a poſſibility of ever coming to an end of the increaſe or addition.

This inexhauſtibility, in the nature of numbers, is, therefore, all that we can diſtinctly comprehend by their infinity; for though we can eaſily conceive that a finite quantity may become greater and greater without end, yet we are not from thence enabled to form any notion of the *ultimatum*, or laſt magnitude, which is incapable of further augmentation.

We cannot, therefore, apply to an infinite ſeries the common notion of a ſum, or a collection of ſeveral particular numbers, which are joined and added together, one after another; for this ſuppoſes that thoſe particulars are all known and determined. But as every ſeries generally obſerves ſome regular law, and continually approaches towards a term or limit, we can eaſily conceive it to be a whole, of its own kind, and that it muſt have a certain real value, whether that value be determinable or not.

Thus, in many ſeries, a number is aſſignable, beyond which no number of its terms can ever reach, or indeed be ever equal to it: but yet may approach to it in ſuch a manner as to want leſs than any given difference. And this we may call the value or ſum of the ſeries; not as being a number found by the common method of addition, but ſuch a limitation of the value of the ſeries, taken in all its infinite capacity, that if it were poſſible to add all the terms together, one after another, the ſum would be equal to that number.

Again, in other ſeries, the value has no limitation; and this may be expreſſed by ſaying, that the ſum of the ſeries is infinitely great; or, which is the ſame thing, that it has no determinate or aſſignable value; but may be carried on to ſuch a length, that its ſum ſhall exceed any given number whatever.

According to the common rule for ſumming up a finite progreſſion of a geometric decreaſing ſeries, where r is the ratio, l the greateſt term, and a the leaſt, the ſum is $(rl-a)\div(r-1)$: and if we ſuppoſe a the leſs extreme, to be actually decreaſed to 0, then the ſum of the whole ſeries will be $rl\div(r-1)$: for it is demonſtrable, that the ſum of no aſſignable number of terms of the ſeries can ever be equal to that quotient; and yet no number leſs than it will ever be equal to the value of the ſeries.

Whatever conſequences, therefore, follow from the ſuppoſition of $rl\div(r-1)$ being the true and adequate value of the ſeries, taken in all its infinite capacity, as if all the parts were actually determined, and added together, they can never be the occaſion of any aſſignable error, in any operation or demonſtration where it is uſed in that ſenſe; becauſe if you ſay that it exceeds that value, it is demonſtrable that this exceſs muſt be leſs than any aſſignable difference, which is, in effect, no difference at all; whence the ſuppoſed error cannot exiſt, and conſequently $rl\div(r-1)$ may be looked upon as expreſſing the adequate and juſt value of the ſeries, continued to infinity.

But we are further ſatisfied of the reaſonableneſs of this doctrine, by finding, in fact, that a finite quantity actually converts into an infinite ſeries, as appears in the caſe of circulating decimals. Thus $\frac{2}{3}$ turned into a decimal is $= 6666$, &c. $=\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}+\frac{6}{10000}$, &c. continued *ad infinitum*. But this is plainly a geometric ſeries, beginning from $\frac{6}{10}$, in the continued ratio of 10 to 1, and the ſum of all its terms, continued to infinity, will evidently be equal to $\frac{2}{3}$, or the number from whence it was originally derived.

And the ſame may be ſhewn of many other ſeries, and of all circulating decimals in general.

PROBLEM I.

Any series being given to find the several orders of differences.

RULE.

1. Take the first term from the second, the second from the third, the third from the fourth, &c. and the remainders will form a new series, called the *first order of differences*.

2. Take the first term of this last series from the second, the second from the third, the third from the fourth, &c. and the remainders will form another new series, called the *second order of differences*.

3. Proceed, in like manner, for the *third*, *fourth*, *fifth*, &c. orders of differences; and so on till they terminate, or are carried as far as is thought necessary.

EXAMPLES.

1. To find the several orders of differences in the series 1, 4, 9, 16, 25, 36, &c.

1,	4,	9,	16,	25,	36,	&c.	
	3,	5,	7,	9,	11,	&c.	*1st diff.*
		2,	2,	2,	2,	&c.	*2d diff.*
			0,	0,	0,	&c.	

2. To find the several orders of differences in the series 1, 8, 27, 64, 125, 216, &c.

1,	8,	27,	64,	125,	216,	&c.	
	7,	19,	37,	61,	91,	&c.	*1st diff.*
		12,	18,	24,	30,	&c.	*2d diff.*
			6,	6,	6,	&c.	*3d diff.*
				0,	0,	&c.	

3. To find the several orders of differences in the series 1, 3, 6, 10, 15, 21, &c.

Ans. 1*st*, 2, 3, 4, 5, *&c.* 2*d* 1, 1, 1, *&c.*

4. To find the several orders of differences, in the series 1, 6, 20, 50, 105, 196, &c.

Ans. 1*st* 5, 12, 30, 45, 91, *&c.* 2*d* 9, 16, 25, 36, *&c.* 3*d* 7, 9, 11, *&c.* 4*th* 2, 2, *&c.*

PROBLEM II.

Any series, a, b, c, d, e, &c. being given to find the first term of the nth. order of differences.

RULE.*

Let δ stand for the first term of the *nth.* differences.

Then will $a-nb+n\times\frac{n-1}{2}c-n\times\frac{n-1}{2}\times\frac{n-2}{3}d+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}e$, &c. to $n+1$ terms $=\delta$, when n is an even number.

And $-a+nb-n\times\frac{n-1}{2}c+n\times\frac{n-1}{2}\times\frac{n-2}{3}d-n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}e$, &c. to $n+1$ terms $=\delta$, when n is an odd number.

* When the several orders happen to be very great, it will be more convenient to take the logarithms of the quantities concerned, whose differences will be smaller; and, when the operation is finished the quantity answering to the last logarithm may be easily found.

EXAMPLES.*

1. Required the first term of the third order of differences of the series 1, 5, 15, 35, 70, &c.

Let $a, b, c, d, e,$ *&c.* $=1, 5, 15, 35, 70,$ *&c. and* $n=3$.

Then $-a+nb-n\times\frac{n-1}{2}c-n\times\frac{n-1}{2}\times\frac{n-2}{3}d=-a+3b-3c+d=-1+15-45+35=4=$ *the first term required.*

2. Required the first term of the fourth order of differences of the series 1, 8, 27, 64, 125, &c.

Let $a, b, c, d, e,$ *&c.* $=1, 8, 27, 64, 125,$ *&c. and* $n=4$.

Then $a-nb+n\times\frac{n-1}{2}c-n\times\frac{n-1}{2}\times\frac{n-2}{3}d+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}e, =a-4b+6c-4d+e=1-32+162-256+125=0$; *so that the first term of the fourth order is* 0.

3. Required the first term of the fifth order of differences of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, &c. *Ans.* $-\frac{1}{16}$.

4. Required the first term of the 8th order of differences of the series 1, 3, 9, 27, 81, &c. *Ans.* 256.

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* The labour in these kind of questions may be often abridged by putting cyphers for some of the terms at the beginning of the series; by which means several of the differences will be equal to 0, and the answer, on that account, obtained in fewer terms.

PROBLEM III.

To find the nth. term of the series, a, b, c, d, e, &c.

RULE.

Let d^{I}, d^{II}, d^{III}, d^{IV}, &c. be the first of the several orders of differences found as in the last problem.

Then will $a+\frac{n-1}{1}d^{I}+\frac{n-1}{1}\times\frac{n-2}{2}d^{II}+\frac{n-1}{1}\times\frac{n-2}{2}\times\frac{n-3}{3}d^{III}+\frac{n-1}{1}\times\frac{n-2}{2}\times\frac{n-3}{3}\times\frac{n-4}{4}d^{IV}$, &c. be $=$*nth. term required.*

EXAMPLES.

1. To find the 12th term of the series, 2, 6, 12, 20, 30, &c.

2,	6,	12,	20,	30,	&c.
	4,	6,	8,	10,	&c.
		2,	2,	2,	&c.
			0,	0,	&c.

Here 4 *and* 2 *are the first terms of the differences.*

Let, therefore, $4=d^{I}$, $2=d^{II}$, *and* $n=12$.

Then $a+\frac{n-1}{1}d^{I}+\frac{n-1}{1}\times\frac{n-2}{2}d^{II}=2+11d^{I}+55d^{II}=2+44+110=156=12th$ *term, or the answer required.*

2. Required the 20th term of the series 1, 3, 6, 10, 15, 21, &c.

1, 3, 6, 10, 15, 21, &c.
2, 3, 4, 5, 6, &c.
1, 1, 1, 1, &c.
0, 0, 0, &c.

Here 2 and 1 are the first terms of the differences. Let, therefore, $2 = d^{\text{I}}$, $1 = d^{\text{II}}$, and $n = 20$,

Then $a + \frac{n-1}{1} d^{\text{I}} + \frac{n-1}{1} \times \frac{n-2}{2} d^{\text{II}} = 1 + 19 d^{\text{I}} + 171 d^{\text{II}} = 1 + 38 + 171 = 210 =$ *20th term required.*

3. Required the 15th term of the series 1, 4, 9, 16, 25, 36, &c. *Ans.* 225.

4. Required the 20th term of the series 1, 8, 27, 64, 125, &c. *Ans.* 8000.

PROBLEM IV.

To find the sum of n terms of the series a, b, c, d, e, &c.

RULE.

Let d^{I}, d^{II}, d^{III}, d^{IV}, &c. be the first of the several orders of differences.

Then will $na + n \times \frac{n-1}{2} d^{\text{I}} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times d^{\text{II}} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times d^{\text{III}} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} d^{\text{IV}}$, &c. = to the sum of n terms of the series.

EXAMPLES.

1. * To find the ſum of n terms of the ſeries 1, 2, 3, 4, 5, 6, &c.

$$\begin{array}{l} 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \&c. \\ \quad 1,\ 1,\ 1,\ 1,\ 1,\ \&c. \\ \qquad 0,\ 0,\ 0,\ 0,\ \&c. \end{array}$$

Here 1 *and* 0 *are the firſt terms of the differences:*
Let therefore $a=1$, $d^{1}=1$, *and* $d^{11}=0$;

Then will $na+n\times\frac{n-1}{2}d^{1}=n+\frac{n^2-n}{2}=\frac{n^2+n}{2}=$ *ſum of* n *terms as required.*

2. To find the ſum of n terms of the ſeries 1^2, 2^2, 3^2, 4^2, 5^2, &c. or 1, 4, 9, 16, 25, &c.

$$\begin{array}{l} 1,\ 4,\ 9,\ 16,\ 25,\ \&c. \\ \quad 3,\ 5,\ 7,\ 9,\ \&c. \\ \qquad 2,\ 2,\ 2,\ \&c. \\ \qquad\quad 0,\ 0,\ \&c. \end{array}$$

Here 3 *and* 2 *are the firſt terms of the differences:*
Let, therefore, $a=1$, $d^{1}=3$, *and* $d^{11}=2$.

Then will $na+n\times\frac{n-1}{2}d^{1}+n\times\frac{n-1}{2}\times\frac{n-2}{3}d^{11}=n+3n+\frac{n-1}{2}+2n\times\frac{n-1}{2}\times\frac{n-2}{3}=\frac{3n^2-n}{2}+\frac{n^3-3n^2+2n}{2}$
$=\frac{n\times(n+1)\times(2n+1)}{6}=$ *sum of n terms, as required.*

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* Any term of a given ſeries, or the ſum of any number of its terms may be accurately determined, when the differences of any order become at laſt equal to each other.

3. To find the sum of n terms of the series 1^3, 2^3, 3^3, 4^3, 5^3, &c. or 1, 8, 27, 64, 125, &c.

$$\begin{array}{rrrrl} 1, & 8, & 27, & 64, & 125, \&c. \\ & 7, & 19, & 37, & 61, \&c. \\ & & 12, & 18, & 24, \&c. \\ & & & 6, & 6, \&c. \\ & & & & 0, \&c. \end{array}$$

Hence the first terms of the differences are 7, 12 *and* 6.
Let, therefore, $a=1$, $d^{\text{I}}=7$, $d^{\text{II}}=12$, *and* $d^{\text{III}}=6$.

Then will $na+n\times\frac{n-1}{2}d^{\text{I}}+n\times\frac{n-1}{2}\times\frac{n-2}{3}d^{\text{II}}$
$+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}d^{\text{III}}=n+7n\times\frac{n-1}{2}+12n$
$\times\frac{n-1}{2}\times\frac{n-2}{3}+6n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}=\frac{7n^2-5n}{2}$
$+2n^3-6n^2+4n+\frac{n^4-6n^3+11n^2-6n}{4}=\frac{n^4+2n^3+n^2}{4}$
$=$ *sum of* n *terms, as required.*

4. To find the sum of n terms of the series 2, 6, 12, 20, 30, &c. *Ans.* $\frac{n\times(n+1)\times(n+2)}{3}$

5. To find the sum of n terms of the series 1, 3, 6, 10, 15, &c. *Ans.* $\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$.

6. To find the sum of n terms of the series 1, 4, 10, 20, 35, &c. *Ans.* $\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}$.

7. To find the sum of n terms of the series 1^4, 2^4, 3^4, 4^4, &c. or 1, 16, 81, 256, &c.

Ans. $\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$.

PROBLEM V.

The series a, b, c, d, e, &c. being given, whose terms are an unit's distance from each other, to find any intermediate term by interpolation.

RULE.

Let x be the diſtance of any term y to be interpolated, and d^{I}, d^{II}, d^{III}, &c. the terms of the differences.

Then will $a+xd^{\text{I}}+x\times\frac{x-1}{2}d^{\text{II}}+x\times\frac{x-1}{2}\times\frac{x-2}{3}d^{\text{III}}+x\times\frac{x-1}{2}\times\frac{x-2}{3}\times\frac{x-3}{4}d^{\text{IV}}$, &c. $=y$.

EXAMPLES.

1. Given the logarithmic ſines of 1° 0′, 1° 1′, 1° 2′ and 1° 3′, to find the ſine of 1° 1′ 40″.

	1° 0′	1° 1′	1° 2′	1° 3′
Sines	8.2418553	8.2490332	8.2560943	8.2630424
		71779	70611	69481
			—1168	—1130
				38

Here the firſt terms of the differences are 71779—1168, *and* 38.

Let, therefore, $x=1°\ 1'\ 40''-1°\ 0'=1'\ 40''=1\frac{2}{3}=$ *diſtance of* y, *the term to be interpolated: and* $d^{\text{I}}=71779$, $d^{\text{II}}=1168$, *and* $d^{\text{III}}=38$.

Then will $y=a+xd^{\text{I}}+x\times\frac{x-1}{2}d^{\text{II}}+x\times\frac{x-1}{2}\times\frac{x-2}{3}d^{\text{III}}=a+\frac{5}{3}d^{\text{I}}+\frac{5}{9}d^{\text{II}}-\frac{5}{81}d^{\text{III}}=8.2418553+$

.0119631—.0000694—.0000002=8.2537533= *sine of* 1° 1′ 40″ *as was required.*

2. Given the series $\frac{1}{50}$, $\frac{1}{51}$, $\frac{1}{52}$, $\frac{1}{53}$, $\frac{1}{54}$, &c. to find the term which stands in the middle between $\frac{1}{51}$ and $\frac{1}{53}$.

Ans. $\frac{2}{105}$.

3. Given the natural tangents of 88°, 54′, 88°, 55′, 88°, 56′, 88°, 57′, 88°, 58′, 88°, 59′; to find the tangent of 88°, 58′, 18″. *Ans.* 55.711144.

PROBLEM VI.

Having given a series of equidistant terms, a, b, c, d, e, &c. whose first differences are small; to find any intermediate term by interpolation.

RULE.*

Find the value of the unknown quantity in the equation which stands against the given number of terms, in the following table, and it will give the term required.

1. $a-b=0$
2. $a-2b+c=0$
3. $a-3b+3c-d=0$
4. $a-4b+6c-4d+e=0$
5. $a-5b+10c-10d+5e-f=0$
6. $a-6b+15c-20d+15e-6f+g=0$
7. $a-nb+n\times\frac{n-1}{2}c-n\times\frac{n-1}{2}\times\frac{n-2}{3}d$, &c. $=0$.

* The more terms are given the more accurately the equation will approximate.

EXAMPLES.

1. Given the logarithms of 101, 102, 104, and 105; to find the logarithm of 103.

Here the number of terms are 4.

Therefore against 4, *in the table, we have* $a-4b+6c-4d+e=o$; *or* $c=\dfrac{4\times(b+d)\times-(a+e)}{6}$

$$\textit{Whence}\begin{cases} a= 2.0043214 \\ b= 2.0086002 \\ d= 2.0170333 \\ e= 2.0211893 \end{cases}$$

$$4\times(b+d)=16{,}1025340$$
$$a+e = 4{,}0255107$$
$$6)12.0770233$$

2.0128372 $=$*log. of* 103, *as required.*

2. Given the cube roots of 45, 46, 47, 48, and 49, to find the cube root of 50. *Ans*. 3.684033.

3. Given the logarithm of 50, 51, 52, 54, 55, and 56, to find the logarithm of 53. *Ans*. 1.7242758695.

PROMISCUOUS EXAMPLES RELATING TO SERIES.

1. To find the ſum (S) of n terms of the ſeries 1, 2, 3, 4, 5, 6, &c.

Firſt, $1+2+3+4+5$, *&c.* .... $n=S$.

And, $n+(n-1)+(n-2)+(n-3)+n-4)$ *&c.* .... $1=S$.

Therefore $(n+1)+(n+1)+(n+1)+(n+1)$, &c. .... $(n+1)=2S$.

And consequently $(n+1)\times n=2S$; *or* $S=\frac{n^2+n}{2}=$ *sum required.*

2. To find the ſum (S) of n terms of the ſeries 1, 3, 5, 7, 9, 11, &c.

Firſt, $1+3+5+7+9$, &c. .... $(2n-1)=S$.

And, $(2n-1)+(2n-3)+(2n-5)+(2n-7)+(2n-9)$, &c. .... $1=S$.

Therefore $2n+2n+2n+2n+2n+$, &c. .... $2n=2S$.

And, consequently, $2n\times n=2S$; *or* $S=\frac{2n\times n}{2}=n^2=$ *sum required.*

3. Required the ſum (S) of n terms of the ſeries $a+(a+d)+(a+2d)+(a+3d)+(a+4d)$, &c.

Firſt, $a+(a+d)+(a+2d)+(a+3d)$, &c. ...... $a+(n-1)\times d=S$.

And $a+(nd-d)+a+(nd-2d)+a+(nd-3d)+a+(nd-4d)$, &c. .... $a=S$.

Therefore $2a+(nd-d)+2a+(nd-d)+2a+(nd-d)$ &c. $2a+(nd-d)=2S$.

And consequently, $(2a+nd-d)\times n=2S$; *or* $S=(2a+nd-d\times\frac{n}{2}=$ *sum required.*

OR THUS:

Firſt, $x+(a+d)+(a+2d)+(a+3d)+(a+4d)$, &c.

$$=\left\{\begin{array}{l}(+1+1+1+1+1, \text{ \&c. } \times d\\(+0+1+2+3+4, \text{ \&c. } \times d\end{array}\right\}=S.$$

But n terms of $1+1+1+1+1$, &c. $=n$.

And n terms of $0+1+2+3+4$, &c. $=\frac{n\times(n+1}{2}$

Therefore $S=na+\frac{n\times(n-1)\times d}{2}=2a+nd-d)\times\frac{n}{2}$ *as before.*

4. To find the sum (S) of n terms of the series 1, x, x^2, x^3, x^4, &c.

First, $1+x+x^2+x^3+x^4$, &c. . . . . $x^{n-1}=S$.
And $x+x^3+x^2+x^4+x^5$, &c. . . . . $x^n=Sx$.
Therefore $-1+x^n=Sx-S$;

Or $S=\frac{x^n-1}{x-1}=$ *sum required.*

And, when x is a proper fraction, the sum of the series, continued ad infinitum, may be found in the same manner,
Thus, $1+x+x^2+x^3+x^4$, &c. $=S$.
And $x+x^2+x^3+x^4+x^5$, &c. $=Sx$.
Therefore $-1=Sx-S$; *or* $S-Sx=1$.

Whence $S=\frac{1}{1-x}=$ *sum of an infinite number of terms as required.*

5. Required the sum (S) of the circulating decimal ·999999, &c. continued *ad infinitum.*

First, $.999999$ &c.$=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\frac{9}{10000}$, &c.

$=9\times:\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}$, &c.$=S$.

Or, $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}$, &c. $=\frac{S}{9}$.

Therefore, $1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}$, &c.$=\frac{10S}{9}$.

And, $1=\frac{10S}{9}-\frac{S}{9}=\frac{9S}{9}$.

Whence $S=1=$ *sum required.*

6. Required the sum (S) of the series $a^2+(a+d)^2+(a+2d)^2+(a+3d)^2+(a+4d)^2$, &c. continued to n terms.

First $a^2=a^2$
$(a+d)^2=x^2+2\times 1ad+1d^2$
$(a+2d)^2=a^2+2\times 2ad+4d^2$
$(a+3d)^2=a^2+2\times 3ad+9d^2$
$(a+4d)^2=a^2+2\times 4ad+16d^2$
&c. *&c.*

$$\textit{Therefore, } S=\left\{\begin{array}{l}\text{Sum of } n \text{ terms of } (1+1+1+1, \&c.) \\ \quad\times a^2 \\ +\ \ldots\ \text{ditto of } (0+1+2+3, \&c.) \\ \quad\times 2ad \\ +\ \ldots\ \text{ditto of } (0+1+4+9, \&c.) \\ \quad\times d^2\end{array}\right.$$

But n *terms of* $1+1+1+1$, *&c.* $=n$.

Ditto of $0+1+2+3$, *&c.* $=\frac{n\times(n-1)}{1\times 2}$

And ditto of $0+1+4+9$, *&c.* $=\frac{n\times(n+1)\times(2n-1)}{1\times 2\times 3}$

Whence $S=n\times a^2+n\times(n-1)\times ad+\frac{n\times(n-1)\times(2n-1)}{1\times 2\times 3}\times d^2=na^2+nad\times(n-1)+\frac{nd^2\times(n-1)\times(2n-1)}{1\times 2\times 3}=$ *sum required.*

7. Required the sum (S) of the series $a^3+(a+d)^3+(a+2d)^3+(a+3d)^3+(a+4d)^3$, *&c.* continued to n terms.

First, $a^3=a^3$
$(a+d)^3=a^3+3\times 1a^2d+3\times 1ad^2+1d^3$
$(a+2d)^3=a^3+3\times 2a^2d+3\times 4ad^2+8d^3$
$(a+3d)^3=a^3+3\times 3a^2d+3\times 9ad^2+27d^3$
$(a+4d)^3=a^3+3\times 4a^2d+3\times 16ad^2+64d^3$
$(a+5d)^3=a^3+3\times 5a^2d+3\times 25ad^3+125d^3$
&c. *&c.*

$$\textit{Therefore, } S = \begin{cases} \textit{Sum of } n \textit{ terms of } (1+1+1+1, \text{\&c.}) \\ \quad \times a^3 \\ + \ldots \textit{ditto of } (0+1+2+3, \text{\&c.} \\ \quad \times 3a^2d. \\ + \ldots \textit{ditto of } (0+1+4+9, \text{\&c.}) \\ \quad \times 3ad^2 \\ + \ldots \textit{ditto of } (0+1+8+27, \text{\&c.}) \\ \quad \times d^3 \end{cases}$$

But n terms of $1+1+1+1$, *&c.* $=n$.

Ditto . . . of $0+1+2+3$, *&c.* $=\dfrac{n\times(n-1)}{1\times 2}$

Ditto . . . of $0+1+4+9$, *&c.* $=\dfrac{n\times(n-1)\times(2n-1)}{1\times 2\times 3}$

Ditto of $0+1+8+27+64+125$, *&c.* $=\dfrac{n^4-2n^3+n^2}{2\times 2}$

Consequently the sum $S=n\times a^3+\dfrac{n\times(n-1)\times 3a^2d}{1\times 2}+$
$\dfrac{n\times(n-1)\times(2n-1)\times 3ad^2}{1\times 2\times 3}+\dfrac{(n^4-2n^3+n^2)\times d^3}{2\times 2}=$
*sum required.**

8. Required the sum (S) of n terms of the series $1+3+7+15+31$, &c.

The terms of this series are evidently equal to 1, $(1+2)$, $(1+2+4)$, $(1+2+4+8)$, *&c. or the successive sums of the geometrical progression* 1, 2, 4, 8, 16, *&c.*

Let, therefore, $a=1$, *and* $r=2$, *and we shall have* $a+ar+ar^2+ar^3+ar^4$, *&c.* $=1+2+4+8+16$, *&c.*

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* For an account of figurative numbers, with the methods of finding their sums, &c. I must refer the learner to Simpson's Algebra, p. 215, where he will find this subject fully explained.

But the sum of 1, 2, 3, 4, *&c. terms of this series are*

1. $\frac{ar-a}{r-1} = (r-1) \times \frac{a}{r-1}$

2. $\frac{ar^2-a}{r-1} = (r^2-1) \times \frac{a}{r-1}$

3. $\frac{ar^3-a}{r-1} = r^3-1) \times \frac{a}{r-1}$

4. $\frac{ar^4-a}{r-1} = r^4-1) \times \frac{a}{r-1}$

&c. *&c.*

Therefore, $S = \frac{a}{r-1} \times \begin{cases} n \text{ terms of } r+r^2+r^3+r^4, \text{ \&c.} \\ -n \text{ terms of } 1+1+1+1, \text{ \&c.} \end{cases}$

But $1+1+1+1+1+1+1$, *&c.* $=n$.

And $r+r^2+r^3+r^4+$, *&c.* $=(r^n-1) \times \frac{r}{r-1}$.

Whence $S=(r^n-1) \times \frac{r}{r-1} - n) \times \frac{r}{r-1}$ = *sum required.*

9. * Required the ſum of (n) terms of the ſeries $\frac{1}{1}+\frac{3}{2}+\frac{7}{4}+\frac{15}{3}+\frac{31}{16}$, &c.

The terms of the series are the successive sums of the geometrical progression $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, *&c.*

Let therefore $a=1$ *and* $r=2$, *then will* $a+\frac{a}{r}+\frac{a}{r^2}+\frac{a}{r^3}+\frac{a}{r^4}+\frac{a}{r^5}+\frac{a}{r^6}$, *&c.* $=\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}$ *&c. continued at pleasure.*

Many queſtions of this kind, as well as ſeveral other things relating to ſeries in general, may be found in Dodſon's Mathematical Repoſitory, which is an excellent analytical performance.

But the sums of 1, 2, 3, 4, *&c. terms of this series are*

$$1\ \frac{(r-1)\times a}{(r-1)\times 1}=(r-1)\times\frac{a}{r-1}$$

$$2.\ \frac{(r^2-1)\times a}{(ar-1)\times r}=(r-\frac{1}{r})\times\frac{a}{r-1}$$

$$3.\ \frac{(r^3-1)\times a}{(r-1)\times r^2}=(r-\frac{1}{r^2})\times\frac{a}{r-1}$$

$$4.\ \frac{(r^4-1)\times a}{(r-1)\times r^3}=(r-\frac{1}{r^3})\times\frac{a}{r-1}$$

&c. *&c.*

Therefore $S=\frac{a}{r-1}\times\begin{cases}n \text{ terms of } r+r+r+r, \text{ \&c.}\\ -n \text{ terms of } \frac{1}{1}+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}, \text{\&c.}\end{cases}$

But $r+r+r+r$, *&c.* $=nr$,

And $\frac{1}{1}+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}$, *&c.* $=\frac{r^n-1}{(r-1)\times r^{n-1}}$,

Whence $S=\frac{a}{r-1}\times(nr-\frac{r^n-1}{(r-1)\times r^{n-1}}=)$ *sum required.*

10. To find the ſum (S) of the infinite ſeries of the reciprocals of the triangular numbers, $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$, &c.

Let $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$, &c. *ad infinitum* $=S$.

Or, $\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{2.3}+\frac{1}{2.5}$, &c. . . . . . $=S$.

Then $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}$, &c. . . . . . $=\frac{S}{2}$

That is, $\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)$ &c. $=\frac{S}{2}$

$$Or, \left\{\begin{array}{l} \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}, \text{ \&c.} \\ -\frac{1}{1}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}, \text{ \&c.} \end{array}\right\} = \frac{S}{2}$$

$$.Whence, \frac{S}{2}=\frac{1}{1}; \text{ or } S=2= \text{ sum required.}$$

11. * To find the sum of n terms of the series $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, &c.

$$Let\ z=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}, \text{ \&c. to } \frac{1}{n}.$$

$$Then\ z-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}, \text{ \&c. to } \frac{1}{n}.$$

$$And\ z-\frac{1}{1}+\frac{1}{n+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}, \text{ \&c. to } \frac{1}{n+1}.$$

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* The figurative numbers, of which the terms of this and several other series are the reciprocals, may be exhibited thus:

Figurative numbers of the	1st. order	are	1, 1, 1, 1, 1, &c.
	2d. order		1, 2, 3, 4, 5, &c.
	3d. order		1, 3, 6, 10, 15, &c.
	4th. order		1, 4, 10, 20, 35, &c.
	5th. order		1, 5, 15, 35, 70, &c.

It may also be remarked, that different series are distinguished by particular names, according to the nature of their terms.

Thus, a series, whose terms continually decrease, is called a diverging series.

And a series, whose terms neither increase nor diminish, is called a neutral series.

Thus: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$, &c. converging; $1-2+3-4+5$, &c diverging: and $1-1+1-1+$, &c. neutral.

Therefore $\frac{1}{1}-\frac{1}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$, *&c. to* $\frac{1}{n}-\frac{1}{n+1}$.

Or, $\frac{n}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$, *&c. to* $\frac{1}{n.(n+1)}$.

Whence, $\frac{2n}{n+1}=\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$, *&c. to* $\frac{2}{n.(n+1)}$.

Or, $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}$, *&c. to* $\frac{2}{n.n+1}$ *(or n terms)* $=\frac{2n}{n+1}=$ *sum of the series, or answer required.*

12. Required the sum of the infinite series $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, &c.

Let $z=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, *&c. ad infinitum.*

Then $z-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ *&c. by transposition.*

And $1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}$, *&c. by subtraction.*

Or, $1-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}$, *&c. by transposition.*

And $\frac{1}{2}=\frac{4}{1.4.3}+\frac{6}{2.9.4}+\frac{8}{3.16.5}$, *&c. by subtraction.*

Or, $\frac{1}{2}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+\frac{2}{4.5.6}$, *&c.*

Whence $\frac{1}{2}\div 2=\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, *ad infinitum.*

But $\frac{1}{2}\div 2=\frac{1}{4}$; *therefore* $\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, *continued to infinity, is equal to* $\frac{1}{4}$, *which is the sum required.*

13. To find the sum of n terms of the series $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, &c.

Let $z=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+$, *&c. to* $\frac{1}{n.(n+1)}$.

Then $z-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}$, *&c. to* $\frac{1}{n.(n+1)}$.

And $z-\frac{1}{2}+\frac{1}{(n+1).(n+2)}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+\frac{1}{6.7}+\frac{1}{7.8}$, *&c. continued to* $\frac{1}{(n+1).(n+2)}$ *terms.*

Therefore $\frac{1}{2}-\frac{1}{(n+1).(n+2)}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}$ (*continued to n terms*) *by subtraction.*

Whence $\frac{1}{4}-\frac{1}{2.(n+1).(n+2)}=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}$, *&c. (continued to n terms) by division.*

And consequently, $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}$ *continued to n terms is* $=\frac{1}{4}-\frac{1}{2.(n+1).(n+2)}=$ *sum required.*

14. Required the sum (S) of the series $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}$, &c. continued *ad infinitum.*

$$Let\ x=\frac{1}{2}\ and\ S=\frac{z}{1+x}$$

Then $\frac{z}{1+x}=x-x^2+x^3-x^4+x^5$, *&c.*

And $z=(1+x)\times(x-x^2+x^3-x^4$, *&c.*)

Whence, by multiplication. $\begin{cases} x-x^2+x^3-x^4+x^5, \text{ \&c.} \\ 1+x \end{cases}$

$$\begin{array}{l} x-x^2+x^3-x^4+x^5, \text{ \&c.} \\ \quad +x^2-x^3+x^4-x^5, \text{ \&c.} \end{array}$$

Whose sum is $=x+0+0+0+0$, &c.

Therefore $z=x$.

And $z-x^2+x^3-x^4+x^5$, &c. $=\dfrac{x}{1+x}$

Or $\dfrac{1}{2}-\dfrac{1}{4}+\dfrac{1}{8}-\dfrac{1}{16}+\dfrac{1}{32}$, &c. $=\dfrac{\frac{1}{2}}{1+\frac{1}{2}}=\dfrac{1}{3}=$ *sum required.*

15. Required the sum of the series $\dfrac{1}{2}+\dfrac{2}{4}+\dfrac{3}{8}+\dfrac{4}{16}+\dfrac{5}{32}$, &c.

Let $x=\dfrac{1}{2}$, *and* $S=\dfrac{z}{(1-x)^2}$.

Then $\dfrac{z}{(1-x)^2}=x+2x^2+3x^3+4x^4+5x^5$, &c.

And $z=(1-x)^2\times(x+2x^2+3x^3+4x^4+5x^5, \text{ \&c.})$

Whence by multiplication $\begin{cases} x+2x^2+3x^3+4x^4, \text{ \&c.} \\ 1-2x+x^2 \end{cases}$

$$\begin{array}{l} x+2x^2+3x^3+4x^4, \text{ \&c.} \\ \quad -2x^2-4x^3-6x^3, \text{ \&c.} \\ \qquad\quad +x^3+2x^4, \text{ \&c.} \end{array}$$

Whose sum is $=x+0 \quad +0 \quad +0 \;\; 0$, &c.

Therefore $x=z$.

And $x+2x^2+3x^3+4x^4+5x^5$, &c. $=\dfrac{x}{(1-x)^2}$

Or $\dfrac{1}{2}+\dfrac{2}{4}+\dfrac{3}{8}+\dfrac{4}{16}+\dfrac{5}{32}$, &c. $=\dfrac{\frac{1}{2}}{(1-\frac{1}{2})^2}=2=$ *sum required.*

16. Required the sum (S) of the infinite series $\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}$, &c.

$$Let\ x=\frac{2}{3},\ and\ \frac{z}{(1-x)^3}=S.$$

Then $\frac{z}{(1-x)^3}=x+4x^2+9x^3+16x^4+25x^5$, &c.

And $z=(1-x)^3\times(x+4x^2+9x^3+16x^4$, &c.$)$ $=x+x^2$, *as will be found by actual multiplication.*

Therefore $x+x^2=z$.

And consequently $x+4x^2+9x^3+16x^4$, &c. $=\frac{x\times(1+x)}{(1-x)^3}$

Or $\frac{1}{3}+\frac{4}{9}+\frac{9}{27}+\frac{16}{81}$, &c. $=\frac{\frac{1}{3}\times(1+\frac{1}{3})}{(1-\frac{1}{3})^3}=\frac{3}{2}=1\frac{1}{2}=$ *sum required.*

17. Required the sum (S) of the infinite series $\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}+\frac{a+3d}{mr^3}$, &c.

$$Let\ x=\frac{1}{4},\ and\ S=\frac{z}{m.(1-x)^2}.$$

Then $\frac{z}{m(1-x)^2}=\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}+\frac{a+3d}{mr^3}$, &c.

Or $\frac{z}{(1-x)^2}=a+\frac{a+d}{r}+\frac{a+2d}{r^2}+\frac{a+3d}{r^3}$, &c.

That is, $\frac{z}{(1-x)^2}=a+(a+d)\times x+(a+2d)\times x^2+(a+3d)\times x^3$, &c.

And $z=(1-x)^2\times(a+(a+d)x+(a+2d)x^2+(a+3d)x^3$, &c.$)=(1-x)\times a+dx$, *as will appear by actual multiplication.*

Therefore $z=(1-x)\times a+dx$.

And consequently $\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}$, &c. $=$

$\frac{(1-x)\times a+dx}{m.(1-x)^2}=$ *sum of the infinite series required.*

EXAMPLES FOR PRACTICE.

1. To find the sum of n terms of the series $a+(a-d)+(a-2d)+(a-3d)+(a-4d)$, &c.

Ans. $\frac{n}{2}\times(2a-\overline{n-1}\times d.)$

2. Required the sum of the infinite series $a+da+d^2a+d^3a+d^4a$, &c. where d is a proper fraction.

Ans. $\frac{a}{1-d}$.

3. Required the sum of the infinite series $1+3x+6x^2+10x^3+15x^4$, &c.

Ans. $\frac{1}{(1-x)^3}$.

4. Required the sum of the infinite series $1+4x+10x^2+20x^3+35x^4$, &c.

Ans. $\frac{1}{(1-x)^4}$

5. Required the sum of the infinite series $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}$, &c.

Ans. $\frac{1}{2}$.

6. Required the sum of 40 terms of the series $(1\times2)+(3\times4)+(5\times6)+(6\times7)+(7\times8)$, &c.

Ans. 22960.

7. To find the sum of the infinite series $1+2^4x+3^4x^2+4^4x^3+5^4x^4$, &c.

Ans. $\frac{1+11x+11x^2+11x^3}{(1-x)^5}$

8. Required the sum of the infinite series $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}$, &c.

Ans. $\frac{1}{18}$.

9. Required the sum of n terms of the series $\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}+\frac{5}{r^5}+\frac{6}{r^6}$, &c.

Ans. $\frac{1}{r-1}-\frac{n}{r(r-1)}$.

10. Required the sum of n terms of the series $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}$, &c.

Ans. $\frac{1}{18}-\frac{1}{3.(n+1).(n+2).(n+3)}$.

11. Required the sum of the series $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}$, &c. $\frac{1}{n\times(3+n)}$.

Ans. $\Sigma=\frac{11}{18}$, $S=\frac{n}{3+3n}+\frac{n}{12+6n}-\frac{n}{17+9n}$.

12. Required the sum of the series $\frac{1}{2.6}+\frac{1}{4.8}+\frac{1}{6.10}$ &c. $\frac{1}{2n(4+2n)}$. *Ans.* $\Sigma=\frac{3}{16}$, $S=\frac{5n+3n^2}{32+48n+16n^2}$.

13. Required the sum of the series $\frac{1}{4.8}-\frac{1}{6.10}+\frac{1}{8.12}-$ &c. $\frac{1}{(2+2n).(6+2n)}$.

Ans. $\Sigma=\frac{1}{48}$, $S=\frac{n}{16+16n}-\frac{n}{36+24n}$.

14. Required the sum of the series $\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}+\frac{1}{12.20}$, &c. $\frac{1}{3n.(4+4n)}$.

Ans. $\Sigma=\frac{1}{12}$, $S=\frac{n}{12+12n}$.

15. * Required the sum of the infinite series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}$, &c.

Ans. .69314718, *&c. or hyp. log. of* 2.

16. Required the sum of the infinite series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}$, &c.

Ans. .78539, &c. *or* $\frac{1}{4}$ *cir. of the circle whose diam. is* 1.

17. Required the sum of the diverging series $\frac{1}{2}-\frac{2}{3}+\frac{3}{4}-\frac{4}{5}+\frac{5}{6}$, &c. *Ans.* .193147, *&c. or* $\frac{1}{2}$ + *hyp. log. of* $2-\frac{1}{2}$.

18. Required the sum of the diverging series $\frac{2^2}{1}-\frac{3^2}{2}+\frac{4^2}{3}-\frac{5^2}{4}$, &c.

Ans. 1.943147, *&c. or* $\frac{5}{4}$ + *hyp. log. of* 2.

19. †Required the sum of the hyper-geometrical series $1-1+2-6+24-120$, &c. or $1-1\text{A}+2\text{B}-3\text{C}+4\text{D}-5\text{E}$, &c. *Ans. near approx. value* = .298174.

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* A great variety of series, of different forms, may be found in other authors; but those which are here given will be sufficient for the learner's practice.

The names of the principal authors, who have written upon this subject, are as follows.

Archimedes; Arabes; D'Alembert; Barrow; Briggs; Nicholas, Daniel, John and James Bernoulli; Fermat; Descartes, Clairaut; Condorcet; Cotes; Dodson; Euler; Emerson; Fagnanus; Le Grange; Goldbach; Gregory; Halley; Harriot; Huddens; Huygens; Hutton; Kepler; Keil; Landen; Maclaurin; De Lagney; Leibnitz; Lorgna; Lucas de Burgo; Manfredi; Monmort; De Moivre; Montano; Nichole; Newton; Oughtred; Riccati; Regnald; Saunderson; Sterling; Slusius; Simpson; Brook Taylor; Varignon; Vieta; Wallis; Waring; &c.

† For an account of these series, with a new method of finding their approximate values, see Hutton's Mathematical Tracts, lately published.

OF LOGARITHMS.*

Logarithms, are numbers ſo contrived and adapted to other numbers, that the ſums and differences of the former ſhall correſpond to, and ſhew, the products and quotients of the latter.

Or, more generally, logarithms are the numerical exponents of ratios; or a ſeries of numbers in arithmetical progreſſion, anſwering to another ſeries of numbers in geometrical progreſſion.

Thus	0,	1,	2,	3,	4,	5,	*Indices, or logarithms.*
	1,	2,	4,	8,	16,	32,	*Geometric progression.*
Or,	0,	1,	2,	3,	4,	5,	*Indices, or logarithms.*
	1,	3,	9,	27,	81,	243,	*Geometric progreſſion.*
Or,	0,	1,	2,	3,	4,	5,	*Ind. or log.*
	1,	10,	100,	1000,	10000,	100000,	*Geog. prog.*

Where it is evident that the ſame indices ſerve equally for any geometric ſeries; and conſequently there may be an endleſs variety of ſyſtems of logarithms to the ſame common numbers, by only changing the ſecond term, 2, 3, or 10, &c. of the geometrical ſeries.

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* The invention of logarithms is the undoubted right of Lord *Naper*, Baron of *Merchiston* in *Scotland*, and is properly conſidered as one of the moſt uſeful and excellent diſcoveries of modern times. A table of theſe numbers was firſt publiſhed by him at Edinburgh, anno 1614, in a treatiſe entitled *Canon Mirificum Logarithmorum*; and, as their great utility and extenſive application were ſufficiently apparent, they were immediately received by all the learned throughout Europe. Mr. *Henry Briggs*, *Savilian* Profeſſor of Geometry at *Oxford*,

It is alſo apparent, from the nature of theſe ſeries, that if any two indices be added together, their ſum will be the index of that number which is equal to the product of the two terms, in the geometric progreſſion, to which thoſe indices belong.

Thus the indices 2 *and* 3, *being added together, are* =5; *and the numbers* 4 *and* 8, *or the terms corresponding with those indices, being multiplied together, are* =32, *which is the number answering to the index* 5.

And, in like manner, if any one index be ſubtracted from another, the difference will be the index of that number, which is equal to the quotient of the two terms to which thoſe indices belong.

Thus the index 6, *minus the index* 4, *is* =2; *and the terms corresponding to those indices are* 64 *and* 16, *whose quotient is* =4; *which is the number answering to the index* 2.

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upon hearing of the diſcovery, ſet out upon a viſit to the noble inventor, and ſoon afterwards they jointly undertook the arduous taſk of computing new tables upon this ſubject, and reducing them to a more convenient form than that which was at firſt thought of. But Lord *Neper* dying before they were finiſhed, the whole burden fell upon Mr. *Briggs*, who with prodigious labour, and great ſkill, made an entire *Canon*, according to the new form, for all numbers from 1 to 20000, and from 9000, to 101000, to 14 places of figures, and publiſhed it at *London* in the year 1624; in a treatiſe entitled *Arithmetica Logarithmica*, with directions for ſupplying the intermediate *chiliads*.

This *Canon* was again publiſhed in *Holland* by *Adrian Vlacq*, anno 1628, together with the logarithms of all the numbers which Mr. *Briggs* has omitted; but he continued them only to 10 places of decimals. Mr. *Briggs* alſo computed the logarithms of the ſines, tangents and ſecants, to every degree, and [illegible] part of a degree of the whole quadrant; and ſubjoined them to the natural ſines, tangents and ſecants, which he had before computed to 15 places of figures.

For the same reason, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power.

Thus the index or logarithm of 4, in the above series, is 2; and if this number be multiplied by 3, the product will be =6; which is the logarithm of 64, or the third power of 4.

And, if the logarithm of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root.

Thus the index or logarithm of 64 is 6; and if this number be divided by 2, the quotient will be =3; which is the logarithm of 8, or the square root of 64.

The logarithms most convenient for practice are such as are adapted to a geometric series increasing

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And these tables together with their construction and use, were first published in the year 1633, after Mr. *Briggs's* death, by Mr. *Henry Gellibrand*, under the title of *Trigonometria Britannica.*

Benjamin Ursinus has also given us a table of logarithms to every 10 seconds. And Mr. *Wolf*, in his *Mathematical Lexicon*, says that one *Van Loser*, had computed them to every single second, but his untimely death prevented their publication.

A great number of other authors have treated on this subject, but as their numbers are frequently inaccurate and incommodiously disposed, they are now generally neglected. The tables in most repute at present, are those of *Gardiner* in 4to, first published in the year 1742, and *Sherwin's* tables in 8vo, first printed in the year 1705, where the logarithms of all numbers may be easily found from 1 to 1000000; and those of the sines, tangents, and secants, to any degree of accuracy required.

Dodson's Antilogarithmic Canon is likewise a very ingenious work, being of great use for finding the numbers answering to any given logarithms.

Since the first publication of this work Mr. *Michael Taylor's* tables have appeared, containing the common logarithms, and the logarithmic sines and tangents to every second of the quadrant.

in a tenfold proportion, as in the laſt of the above forms; and are thoſe which are to be found, at preſent, in moſt of the common tables upon this ſubject.

The diſtinguiſhing mark of this ſyſtem of logarithms, is, that the index, or logarithm, of 1 is 0; that of 10, 1; that of 100, 2; that of 1000, 3, &c. And in decimals the logarithm of .1 is —1; that of .01, —2; that of .001, —3, &c.

From whence it follows that the logarithm of any number between 1 and 10 muſt be 0 and ſome fractional parts, and that of a number between 10 and 100 will be 1 and ſome fractional parts; and ſo on for any other number whatever.

And ſince the integral part of a logarithm is always thus readily found, it is uſually called the *index*, or *characteriſtic*; and is commonly omitted in the tables; being left to be ſupplied by the operator himſelf, as occaſion requires.

OF THE MAKING

OF LOGARITHMS.

Whatever arithmetical progreſſion we apply to a geometrical one, the terms of it are logarithms only to that ſeries to which we apply them, and anſwer the end propoſed only for thoſe particular numbers; ſo that if we had logarithms adapted only to particular geometrical ſeries, they would be but of little uſe. The great end and deſign of theſe numbers is the eaſe and expedition which they afford in long calculations, by ſaving the laborious work of *multiplication*, *diviſion*, and the *extraction of roots*; but this end would never

be completely anſwered, unleſs logarithms could be adapted to the whole ſyſtem of numbers, 1, 2, 3, 4, &c. And as here lies the chief excellence and merit of the contrivance, ſo alſo the difficulty. For the natural ſyſtem of numbers, 1, 2, 3, 4, &c. being an arithmetical, and not a geometrical ſeries, ſeems rather fit to be made logarithms of, than to have logarithms applied to it. But this difficulty may be eaſily removed, by conſidering,

That though the whole ſyſtem of natural numbers, 1, 2, 3, 4, &c. is not in geometrical progreſſion, and cannot, by any means, be made to agree with ſuch a ſeries, yet they may be brought ſo near it, as to be within any aſſignable degree of approximation; which may be conceived, in general, thus: ſuppoſe a fraction indefinitely ſmall to be repreſented by x, and a geometrical ſeries ariſing from 1, in the ratio of 1 to $1+x$, to be 1, $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, $(1+x)^4$, &c. Then muſt ſome of theſe terms come indefinitely near to all the natural numbers 1, 2, 3, 4, &c.; becauſe, amongſt numbers which ariſe by extremely ſmall increments, ſome of them muſt exceed, or fall ſhort, of any determinate number, by an indefinitely little exceſs or defect.

If, therefore, in the places of the terms of this ſeries, which approach indefinitely near to any of the natural numbers, we ſuppoſe theſe natural numbers themſelves to be ſubſtituted, then will this ſeries be in geometrical progreſſion, to an exactneſs which may be called *indefinite;* becauſe the approximation of its terms to the natural numbers can never end, but goes on *in infinitum.*

And ſince this imagined geometric ſeries comprehends, indefinitely near, the whole ſyſtem of natural numbers, 1, 2, 3, 4, &c. ſo the indices of its terms comprehend a whole ſyſtem of logarithms, which are

adapted to this syſtem of numbers, and may be extended to any length we pleaſe. For though the natural ſyſtem of numbers make not, by themſelves a complete geometrical ſeries, yet they are conceived as a part of ſuch a ſeries, and their logarithms are the indices of their diſtances from unity in that ſeries; or, more generally, they are the correſponding terms of an arithmetical ſeries applied to that geometrical one.

But, again, it muſt be obſerved, that an indefinitely ſmall fraction cannot be aſſigned; and, therefore, in the actual conſtruction of logarithms, we muſt be contented with a determinate degree of approximation. Whence, according as we take x, in the ſeries 1, $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, $(1+x)^4$, &c. the approximation of its terms to the natural numbers will be in different degrees of exactneſs: for the leſs x is, the nearer will be the approximation; but then the more are the number of involutions of $1+x$, neceſſary to come within any determinate degree of nearneſs to the natural number aſſigned.

Thus then we may conceive the poſſibility of making logarithms to all the natural numbers, 1, 2, 3, 4, &c. to any determinate degree of exactneſs; viz. by aſſigning a very ſmall fraction for x, and actually raiſing a ſeries, in the ratio of 1 to $1+x$, and taking for the natural numbers ſuch terms of that ſeries as are neareſt to them, and their indices for the logarithms. But then, to conſtruct logarithms in this manner, to ſuch an extent of numbers, and degree of exactneſs, as would be neceſſary to make them of any conſiderable uſe, is next to impoſſible, becauſe of the almoſt infinite labour and time it would require. This, however, is an introduction for underſtanding the method of the *noble inventor*, who undoubt-

edly first took the hint of making logarithms from the consideration of the indices of a geometrical series; and by means of the principles and known properties of these progressions he first formed his tables, and adapted them to the practical purposes intended.

PROBLEM I.

To find the logarithm of any of the natural numbers, 1, 2, 3, 4, *&c. according to the method of* NEPER.

RULE.*

1. Take the geometrical series, 1, 10, 100, 1000, 10000, &c. and apply to it the arithmetical series 1, 2, 3, 4, &c. as logarithms.

2. Find a geometric mean between 1 and 10, 10 and 100, or any other two adjacent terms of the series betwixt which the number proposed lies.

3. Between the mean, thus found, and the nearest extreme, find another geometrical mean, in the same manner; and so on, till you are arrived within the proposed limit of the number whose logarithm is sought.

4. Find as many arithmetical means, in the same order as you found the geometrical ones, and the last of these will be the logarithm answering to the number required.

* The reader who wishes to inform himself more particularly concerning the history, nature, and construction of logarithms, may consult Hutton's Mathematical Tables, lately published, where he will find his curiosity amply gratified.

EXAMPLES.

Let it be required to find the logarithm of 9.

Here the numbers between which 9 *lies are* 1 *and* 10.

First, then, the log. of 10 *is* 1, *and the log. of* 1 *is* 0; *therefore* $\frac{1+0}{2}=.5$ *is the arithmetical mean, and* $\sqrt{(1\times10)}=\sqrt{10}=3{,}1622777$ = *geometric mean: whence the logarithm of* 3,1622777 *is* .5.

Secondly, the log. of 10 *is* 1, *and the log. of* 3.1622777 *is* .5; *therefore* $\frac{1+.5}{2}=.75$ = *arithmetical mean, and* $\sqrt{(10\times3.1622777)}=5.6234132$ = *geometric mean: whence the log. of* 5.6234132 *is* .75.

Thirdly, the log. of 10 *is* 1, *and the log. of* 5.6234132 *is* .75; *therefore* $\frac{1+.75}{2}=.875$= *arithmetical mean, and* $\sqrt{(10.\times5.6234132)}=7.4989421$= *geometric mean: whence the log. of* 7.4989421 *is* .875.

Fourthly, the log. of 10 *is* 1, *and the log. of* 7.4989421 *is* .875; *therefore* $\frac{1+.875}{2}=.9375$ = *arithmetical mean, and* $\sqrt{(10\times7.4989421)}=8.6596431$= *geometric mean: whence the log. of* 8.6596431 *is* .9375.

Fifthly, the log. of 10 *is* 1, *and the log. of* 8.6596431 *is* .9375; *therefore* $\frac{1+.9375}{2}=.96875$ = *arithmetical mean, and* $\sqrt{(10\times8.6596431)}=9.3057204$ = *geometric mean: whence the log. of* 9.3057204 *is* .96875.

Sixthly, the log. of 8.6596431 *is* .9375, *and the log. of* 9.3057204 *is* .96875; *therefore* $\frac{.9375+.96875}{2}=$ .953125= *arith. mean, and* $\sqrt{(8.6596431\times 9.3057204)}$ =8.9768713= *geometric mean: whence the log. of* 8.9768713 *is* .953125.

And, proceeding in this manner, after 25 *extractions, the logarithm of* 8.9999998 *will be found to be* .9542425; *which may be taken for the logarithm of* 9, *because it differs from it only by* $\frac{1}{10000000}$, *and is therefore sufficiently exact for all practical purposes.*

And in the same manner the logarithms of almost all the prime numbers were found; a work so incredibly laborious, that the unremitted industry of several years was scarcely sufficient for its accomplishment.

PROBLEM II.

To determine the hyperbolic logarithm (L) *of any given number* (N).

The hyperbolic logarithm of any number is the index of that term of the logarithmic progreſſion, which agrees with the propoſed number multiplied by the exceſs of the common ratio above unity.

Let, therefore, $(1+x)^n$ be that term of the logarithmical progreſſion, 1, $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, $(1+x)^4$, &c. which is equal to the required number (N).

Then will $(1+x)^n=N$, and $1+x=N^{\frac{1}{n}}$: and if $1+y$ be put $=N$, and $m=\frac{1}{n}$, we ſhall have $1+x$ $=N^{\frac{1}{n}}=(1+y)^m=1+my+m\times\frac{m-1}{2}y^2+m\times\frac{m-1}{2}\times\frac{m-2}{3}y^3$, &c.

And, consequently, $x=my+m\times\frac{m-1}{2}y^2+m\times\frac{m-2}{2}\times\frac{m-2}{3}y^3$, &c. where m being rejected in the factors $m-1$, $m-2$, $m-3$, &c. being indefinitely small in comparison of 1, 2, 3, &c. the equation will become $x=my-\frac{my^2}{2}+\frac{my^3}{3}-\frac{my^4}{4}$, &c.

Hence $\frac{x}{m}$ $(nx=\text{L})=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{3}+\frac{y^5}{5}$, &c. = hyperbolic logarithm of N, as was required.

PROBLEM III.

The hyperbolic logarithm (L) *of a number being given, to find the number* (N) *itself, which answers to it.*

Let $(1+x)^n$ be that term of the logarithmic progression, 1, $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, $(1+x)^4$, &c. which is equal to the required number N.

Then, because $(1+x)^n$ is universally $=1+nx+n\times\frac{n-1}{2}x^2+n\times\frac{n-1}{2}\times\frac{n-2}{3}x^3$, &c. we shall have $1+nx+n\times\frac{n-1}{2}x^2+n\times\frac{n-1}{2}\times\frac{n-2}{3}x^3$, &c. = N.

But since n, from the nature of the logarithms, is here supposed indefinitely great, it is evident that the numbers connected to it by the sign — may all be rejected, as far as any assigned number of terms.

For as 1, 2, 3, &c. are indefinitely small in comparison to n, the rejecting of those numbers can very little affect the values to which they belong.

If, therefore, 1, 2, 3, &c. be thrown out of the factors $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, &c. we shall have $1+nx+\frac{n^2x^2}{2}+\frac{n^3x^3}{2.3}+\frac{n^4x^4}{2.3.4}$, &c. $=N$.

But nx ($=L$) is the hyperbolic logarithm of $(1+x)^n$, or N, by what has been before specified; and therefore $1+L+\frac{L^2}{2}+\frac{L^3}{2.3}+\frac{L^4}{2.3.4}$, &c. $=N=$ number required.

PROBLEM IV.

To determine the hyperbolic logarithm (L) *of any given number* (N,) *by an universally converging series.*

The series $y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}$, &c. is the most easy and natural that can be obtained; but, in determining the logarithms of large numbers, it is but of little use, since, in such cases, it diverges instead of converging.

Let, therefore, the number whose logarithm you would find be denoted by $\frac{1}{1-y}$, and also let $(1+x)^n$ be the term of the logarithmic progression agreeing with the proposed number.

Then $(1+x)^n=\frac{1}{1-y}$; or $1+x=\frac{1}{(1-y)^{\frac{1}{n}}}=$ $(1-y)^{-\frac{1}{n}}=(1-y)^m$ (by putting $m=-\frac{1}{n}$) $=1-my+m\times\frac{m-1}{2}y^2-m\times\frac{m-1}{2}\times\frac{m-2}{3}+y^3$, &c.

Whence $y+\frac{y^2}{2}+\frac{y^3}{3}+\frac{y^4}{4}$, &c. $=-\frac{x}{m}=nx=$ hyperbolic logarithm of $\frac{1}{1-y}$; which ſeries, it is manifeſt, will conſtantly converge, let the value of $\frac{1}{1-y}$ be ever ſo great; becauſe y will always be leſs than unity.

But it is to be obſerved that this ſeries, except in its ſigns, has exactly the ſame form with that above found for the logarithm of $1+y$, and that, if both of them be added together, the ſeries $2y+\frac{2y^3}{3}+\frac{2y^5}{5}+\frac{2y^7}{7}$, &c. thence ariſing, will be more ſimple than either of them, as one half of the terms will, in that caſe, be entirely deſtroyed.

Since, therefore, the ſum of the logarithms of any two numbers is equal to the logarithm of the product of thoſe numbers, it is manifeſt that $2x+\frac{2x^3}{3}+\frac{2x^5}{5}$ &c. will truly expreſs the logarithm of $\frac{1+x}{1-x}$; which ſeries converges ſtill faſter than $x+\frac{x^2}{2}+\frac{x^3}{3}$, &c. not only becauſe the even powers are here deſtroyed, but becauſe x, in finding the logarithm of any given number (N) will have a leſs value.

And, in order to determine what this value of x muſt be, make $\frac{1+x}{1-x}=N$, and then x will be found $=\frac{N-1}{N+1}$; but if the quantity propoſed $\frac{P}{Q}$ be a fraction, inſtead of a whole number, make $\frac{1+x}{1-x}=\frac{P}{Q}$.

and we shall have $x=\frac{P-Q}{P+Q}$; and either of these values being substituted in the foregoing series $2x+\frac{2x^3}{3}+\frac{2x^3}{5}$, &c. will give the hyperbolic logarithm of the number required.

Now, by finding *Neper's* logarithm of any number, according to the foregoing method, *Briggs's*, or the common logarithm of the same number, may be found as follows:

Briggs's logarithm of any number is, to *Neper's* logarithm of the same number, as *Briggs's* logarithm of 10 is to *Neper's* logarithm of 10.

But *Briggs's* logarithm of 10 is 1, and *Neper's* logarithm of 10 is 2.302585093, whence if *Briggs's*, or the common logarithm of any number, be denoted by C. L. and *Neper's* or the hyperbolic logarithm of the same number, by H. L. we shall have $2.302585093 : 1 :: \text{H.L.} : \text{C.L.}$; or $\text{H.L.} \times \frac{1}{2.302585093} = \text{H.L.} \times .4342944819 = \text{C. L.}$ as was required.*

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* There are, besides these, many other ingenious methods, which later writers have discovered, for finding the logarithms of numbers in a much easier way than the original inventor; but, as they cannot be understood without a knowledge of some of the higher branches of the mathematics, I have thought proper to omit them, and must beg leave to refer the reader to those works which are written expressly upon the subject.

It would likewise much exceed the limits of this compendium, to point out all the peculiar artifices that are made use of for constructing an entire table of these numbers; such as those of *Gardiner*, *Sherwin*, and others, who have treated on this subject; but any information of this kind, which the learner may wish to obtain, may be found in Hutton's Tables, before mentioned.

OF THE METHOD OF USING

A TABLE OF LOGARITHMS.

Having explained the method of making a table of the logarithms of numbers greater than unity, the next thing to be done is, to ſhew how the logarithms of fractional quantities may be found. And, in order to this, it may be obſerved, that as we have hitherto ſuppoſed a geometric ſeries to increaſe from an unit on the right hand, ſo we may now ſuppoſe it to decreaſe from an unit towards the left; and the indices, in this caſe, being made negative, will ſtill exhibit the logarithms of the terms to which they belong.

Thus Log. -3 -2 -1 0 $+1$ $+2$ $+3$, *&c.*

Num. $\frac{1}{1000}$, $\frac{1}{100}$, $\frac{1}{10}$, 1 10 100 1000, *&c.*

Whence $+1$ *is the logarithm of* 10, *and* -1, *the logarithm of* $\frac{1}{10}$; $+2$ *the logarithm of* 100, *and* -2 *the logarithm of* $\frac{1}{100}$, *&c.*

And from hence it appears that all numbers, conſiſting of the ſame figures, whether they be integral, fractional, or mixed, will have the decimal parts of their logarithms the ſame.

It will be ſufficient to obſerve here, that the logarithms of all the prime numbers being had, thoſe of the compoſite numbers may be found only by means of addition and ſubtraction. Thus, L.4 2L.2; L.10 L.2 . L.5; L.5 = L.10 − L.2; L.6 = L.2 , L.3, and ſo on for any other of theſe numbers.

In like manner the log. of $a \times b$ = L.a + L.b; the log. of $a \div b$ = L.a − L.b; the log. of r^n = nL.r, and the log. of $r^{\frac{1}{n}} = \frac{1}{n}$L.$r$.

Thus the logarithm of 5874 being 3.7689339, the logarithm of $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$ *&c. part of it will be as follows.*

Num.	Logarithms.
5874	3.7689339
587.4	2.7689339
58.74	1.7689339
5.874	0.7689339
.5874	—1.7689339
.05874	—2.7689339
.005874	—3.7689339

From this it alſo appears, that the *index*, or *characteriſtic*, of any logarithm, is always one leſs than the number of figures which the natural number conſiſts of: and this index is conſtantly to be placed on the left hand of the decimal part of the logarithm.

When there are integers in the given number, the index is always affirmative; but when there are no integers, the index is negative, and is to be marked by a line drawn before it, like a negative quantity in algebra.

Thus a number having 1, 2, 3, 4, 5, *&c. integer places,*

The index of its log. is 0, 1, 2, 3, 4, *&c.*

And a fraction having a digit in the place of primes, seconds, thirds, fourths, &c.

The index of its logarithm will be —1, —2, —3, —4, *&c.*

It may alſo be obſerved, that though the indices of fractional quantities are negative, yet the decimal parts of their logarithms are always affirmative; and all operations are to be performed by them in the ſame manner as by negative and affirmative quantities in algebra.

In taking out of a table the logarithm of any number, not exceeding 100000, we have the decimal part by inspection; and if to this the proper characteristic be affixed, it will give the complete logarithm required.

But if the number, whose logarithm is required, be above 100000, then find the logarithm of the two nearest numbers to it that can be found in the table, and say, as their difference : the difference of their logarithms :: the difference of the nearest number and that whose logarithm is required : the difference of their logarithms, *nearly*; and this difference being added to, or subtracted from, the nearest logarithm, according as it is greater or less than the required one, will give the logarithm required *nearly*.

Thus, let it be required to find the logarithm of 367182.*

The decimal part of 3671 *is, by the table* 5647844; *and of* 3672 *is* .5649027 :

$$\therefore \textit{The log. of} \left\{\begin{matrix} 367100 \textit{ is } 5.5647844 \\ 367200 \textit{ is } 5.5649027 \end{matrix}\right\}$$

$$\begin{array}{rl} \textit{Their diff.}\ 100 & .0001183\ \textit{diff.} \\ \textit{Nearest No} & \left\{\begin{matrix} 367200 \\ 367182 \end{matrix}\right. \\ \textit{Given No} & \\ & \overline{18\ \textit{diff.}} \end{array}$$

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* This method, being founded on the supposition that the logarithms of all numbers between 367100 and 367200, increase or decrease, equally, according to their distance from 367100 or 367200, is not strictly true, but nearly so; and the greater any numbers are, with respect to their difference, the nearer will those differences be proportional. And, therefore, though this method will not give the exact logarithm, yet it will be a very near approximation, and is sufficiently exact for most practical purposes.

Therefore 100 : .0001183 :: 18 : .0000212.

And 5.5649027—.0000212=5.5648815= *logarithm of* 367182 *nearly*.

If the number consists both of integers and fractions, or is entirely fractional, find the decimal part of the logarithm as if all its figures were integral; and this, being prefixed to the proper characteristic, will give the logarithm required.

And if the given number is a proper fraction, subtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm sought; which, being that of a decimal fraction, must always have a negative index.

And, if it is a mixed number, reduce it to an improper fraction, and find the difference of the logarithms of the numerator and denominator, in the same manner as before.

In finding the number answering to any given logarithm, the index, if *affirmative*, will always shew how many integral places the required number consists of; and, if *negative*, in what place of decimals the first, or significant figure, stands; so that, if the logarithm can be found in the table, the number answering to it will always be had by inspection.

But, if the logarithm cannot be exactly found in the table, find the next greater, and the next less, and then say, As the difference of these two logarithms : the difference of the numbers answering to them :: the difference of the given logarithm and the nearest tabular logarithm : a fourth number; which added to, or subtracted from, the natural number answering to the nearest tabular logarithm, according as that logarithm is less or greater than the given one, will give the number required, *nearly*.

Thus let it be required to find the natural number answering to the logarithm 5.5648815.

The next less and greater logarithms, in the table, are

5.5647844	*The numbers*	367100
5.5649027	*answering*	367200
Their diff. .0001183		100

And 5.5649027—5.5648815=.0000212.

Therefore .0001183 : 100 :: .0000212 : 18 *nearly.*

Whence 367200—18=377182= *number required.**

MULTIPLICATION BY LOGARITHMS.

RULE.

Add the logarithms of the factors together, and their sum will be the logarithm of the product required.

Observing to add what is to be carried from the decimal part of the logarithm to the sum of the affirmative indices:

And that the difference between the affirmative and negative indices is to be taken for the index to the logarithm of the product.

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* Directions at large, for the using of logarithms, may be found in most of the common tables upon this subject.—*Sherwin's* Mathematical tables, of the *Edition* 1741 or 1742, are reckoned the most correct and convenient, for practical purposes, of any now extant, except those of Dr. Hutton, lately published; which besides their accuracy, are much better arranged; and, in the two first degrees the sines, &c. are given to every second. He has also a table of hyperbolic logarithms, and several others equally useful.

EXAMPLES.

1. Let the number 256 be multipled by 4.
The log. of 256 = 2.4082400
The log. of 4 = 0.6020600

The product =1024 ..3.0103000

2. Let the number 8.5 be multiplied by 10.
The log. of 8.5 = 0.9294189
The log. of 10 = 1.0000000

The product =85....1.9294189

3. Let the number 46.75 be multiplied by .3275.
The log. of 46.75 = 1.6697816
The log. of .3275 =—1.5152113

The product =15.31.... 1.1849925

4. Multiply 3.768, 2.053, and .007693 continually together.
The log. of 3.768 = 0.5761109
The log. of 2.053 = 0.3123889
The log. of.007693 =—3 8860997

The product =.059511... —2.7745955

5. Multiply .5, .4, and .12, continually together.
The log. of .5=—1.6989700
The log. of .4=—1.6020600
The log. of .12=—1.0791812

The product =.024...—2.3802112

DIVISION BY LOGARITHMS.

RULE.

From the logarithm of the dividend ſubtract the logarithm of the diviſor, and the number agreeing to the remainder will be the quotient required.

But obſerve to change the index of the diviſor from negative to affirmative, or from affirmative to negative, and then the difference of the affirmative indices muſt be taken for the index to the logarithm of the quotient.

And, alſo, when an unit is borrowed, in the left hand place of the decimal part of the logarithm, add it to the index of the diviſor; but if it be negative, ſubtract it: and let the index ariſing from thence be changed and worked with as before.

EXAMPLES.

1. Let the number 56 be divided by the number 4.

The log. of	56 =	1.7481880
The log. of	4 =	0.6020600
The quotient =	14	1.1461280

2. Let the number 50.75 be divided by the number .25.

The log. of	50.75 =	1.7054360
The log. of	.25 =	=—1.3979400
The quotient =	203	2.3074960

3. Let the number .24 be divided by the number 80.

The log. of	.24 =	—1.3802112
The log. of	80 =	1.9030900
The quotient	.003	—3.4771212

4. Let the number .01265 be divided by the number .35

The log. of	.01265 =	—2.1020905
The log. of	.35 =	—1.7403627
The quotient	=.023. . . .	—2.3617278

INVOLUTION BY LOGARITHMS.

RULE.*

1. Seek the logarithm of the given number in the table.

2. Multiply the logarithm, thus found, by the index of the propoſed power.

3. Find the number correſponding to the product, and it will be the power required.

Note. In multiplying a logarithm with a negative index, by any affirmative number, the product will always be negative:

But what is to be carried from the decimal part of the logarithm will always be affirmative;

And therefore their difference will be the index of the product; and is conſtantly to be made of the ſame kind with the greater,

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* The rule of proportion is performed by adding the logarithms of the two laſt terms, and ſubtracting the logarithm of the firſt.

EXAMPLES.

1. Required the second power of the number 3.874.

The log. of	3.874 =	0.5881596
The index =		2
The power = 15.01		1.1763192

2. Required the third power of the number 2.768.

The log. of	2.768 =	0.4421661
The index =		3
The power = 21.21		1.3264983

3. Required the third power of the number .7916.

The log. of	.7916 =	—1.8985058
The index =		3
The power = .4961		—1.6955174

4. Required the twelfth power of the number 1.539.

The log. of	1.539 =	0.1872386
The index =		12
The power = 176.6		2.2468632

5. Required the 20th power of 1.05.

Ans. 2.6533, &c.

6. Required the 100th power of 1.05.

Ans. 131.50, &c.

EVOLUTION BY LOGARITHMS.

RULE.

1. Seek the logarithm of the given number in the table.

2. Divide the logarithm, thus found, by the denominator of the index of the root propoſed.

3. Find the number correſponding to this quotient, and it will be the root required.

Note. When the index of the logarithm, to be divided, is negative, and does not exactly contain the diviſor, increaſe it by ſuch a number as will make it exactly diviſible, and carry the units borrowed, as ſo many tens, to the left hand place of the decimal, and then divide as in whole numbers.

EXAMPLES.

1. Required the ſquare root of the number 225.

The log. of 225=2.3521825

Therefore 2)2.3521825

The root = 15 . . . 1.1760912

2. Required the ſquare root of the number 1501.

The log. of 1501=3.1763807

Therefore 2)3.1763807

The root = 38.74 . . . . 1.5881903

3. What is the cube root of the number .166375?

The log. of .166375=—1.2210881
Therefore 3)—1.2210881

The root =.55 . . . . . —1.7403527

4. What is the square root of the number .08162?

The log. of .08162 =—2.9117966
Therefore 2)—2.9117966

The root =.2857 . . . . —1.4558983

5. What is the twelfth root the number 176.6?

The log. of 176.6=2.2469907
Therefore 12)2.2469907

The root =1.539 1872492

MISCELLANEOUS QUESTIONS.

1. A person being asked what o'clock it was, answered, that it was between 8 and 9, and that the hour and minute hands were exactly together; what was the time?

Ans. $8^h : 43' : 38\frac{2}{11}''$.

2. Divide the number 50 into two such parts, that $\frac{3}{4}$ of one part, added to $\frac{5}{6}$ of the other may make 40.

Ans. 20 *and* 30.

3. What two numbers are those, whose difference is 12, and their squares equal to each other?

Ans. +6 *and* —6.

4. There is a certain number, consisting of two places which is equal to the difference of the squares of its digits; and if 36 be added to it, the digits will be inverted; quære the number? *Ans.* 48.

5. Given $x^3+y^2=31$, and $y^3+x^2=17$; to find x and y. *Ans.* $x=3$ *and* $y=2$.

6. Given $y^2-xy=666$, and $x^2+xy=406$; to find x and y. *Ans.* $x=7$, *and* $y=9$.

7. Given the ſum of three numbers, in harmonical proportion, $=26$, and their continued product $=576$; to find the numbers. *Ans.* 12, 8, *and* 6.

8. What two numbers are thoſe, whoſe difference, ſum, and product, are to each other as the numbers 2, 3, and 5, reſpectively? *Ans.* 2 *and* 10.

9. * To find that number whoſe cube being ſubtracted from its ſquare, ſhall leave the greateſt remainder poſſible? *Ans.* $\frac{2}{3}$.

10. It is required to find the leaſt 3 whole numbers, ſo that $\frac{1}{8}$ of the firſt, $\frac{5}{14}$ of the ſecond, and $\frac{7}{18}$ of the third, ſhall be all equal to each other.

Ans. 280, 294 *and* 300.

11. Given $zx^3+xz^3=290$, and $x^4+z^4=641$; to find x and z. *Ans.* $x=5$, *and* $z=2$.

12. Given the ſum of three numbers in continued geometrical progreſſion $=39$, and the ſum of their ſquares $=819$; to find the numbers. *Ans.* 3, 9, 27.

13. Required the leaſt number of weights, and the weight of each, that will weigh from one pound to 29 hundred weight.

Ans. 1, 3, 9, 27, 81, 243, 729, *and* 2187.

14. Required two numbers ſuch, that their ſum ſhall be equal both to their product and the difference of their ſquares. *Ans.* 2.618034 *and* 1.618034.

15. It is required to find the leaſt 4 affirmative integers ſuch, that the ſquare of the greateſt may be equal to the ſum of the ſquares of the other three.

Ans. 3, 4, 12, *and* 13.

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* This is properly a queſtion in fluxions, but it is anſwered algebraically by Mr. Emerſon, as well as ſeveral others of the ſame nature.

16. If money be lent, at three per cent,
To those who choose to borrow,
In what time shall I be worth a pound,
If I lend a crown to morrow?

Ans. 46.90036 *years allowing comp. int.*

17. Required the two least nonquadrate numbers, x and y, such, that x^2+y^2 and x^3+y^3 shall be both square numbers. *Ans.* $x=364$, *and* $y=273$.

18. There are three numbers in geometrical proportion such, that, if the mean be subtracted from the sum of the two extremes, the remainder multiplied by the sum of the said two extremes will be $9\frac{1}{2}$; but if that remainder be multiplied by the sum of all the three numbers, the product will be 133; it is required to find the three numbers by a simple equation. *Ans.* 4, 6, *and* 9.

19. To determine two numbers whose sum shall be a cube, but their product and quotients squares.

Ans. 4 *and* 14, 100 *and* 25, 900 *and* 100.

20. Required that arithmetical progression whose number of terms is 10, sum of the terms 185, and the sum of the cubes of the terms 104525.

Ans. 5, 8, 11, 14, 17, 20, 23, 26, 29, 32.

21. To divide a given number (N) into 4 such parts, that if any other number (n) be added to the first part, deducted from the second, multiplied by the third, and the fourth part divided by it, the sum, difference, product and quotient, shall be all equal to each other.

$$Ans.\ \frac{Nn}{(n+1)^2}-n,\ \frac{Nn}{(n+1)^2}+n,\ \frac{N}{(n+1)^2}\ and\ \frac{Nn\times n}{(n+1)^2}.$$

22. Given $xy+y^2x=512500$, and $x^2y-y^2x=5500$; to find x and y. *Ans.* $x=25$, *and* $y=20$.

23. Given $x+y+z=6$, $xy+xz+yz=11$, and $xyz=6$; to find x, y, and z. *Ans.* $x=3$, $y=1$, *and* $z=2$.

24. To find two numbers in the ratio of 5 to 7, which being respectively divided by 9 and 13, shall leave 3 and 8 for remainders. *Ans.* 210 *and* 294.

25. To find three numbers such, that $\frac{1}{2}$ the first, $\frac{1}{3}$ the second, and $\frac{1}{4}$ of the third, shall be $=62$; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third $=47$; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third $=38$.

Ans. 24, 60 *and* 120.

26. A, B, and C, are to share 100,000 pounds between them, in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, respectively; but C's part being lost by death, it is required to divide the whole sum properly between the other two.

Ans. A*'s share is* $57142\frac{282}{329}$*l. and* B*'s* $42857\frac{47}{329}$*l.*

27. To find four numbers, x, y, z, and w, having the product of every three given; viz. $xyz=231$, $xyw=420$, $yzw=1540$, and $xzw=660$.

Ans. $x=3$, $y=7$, $z=11$, *and* $w=20$.

28. To find four numbers in geometrical proportion, whose sum is 15, and the sum of their squares 85.

Ans. 1, 2, 4, 8.

29. To find three numbers, x, y, and z, when the product of each by the sum of the other two are given; viz. $x\times(y+z)=48$, $y\times(x+z)=39$, and $z\times(x+y)=63$. *Ans.* $x=4$, $y=3$, *and* $z=9$.

30. What number is that, which, being any how divided, the square of one part, when added to the other part, shall always be a square number? *Ans.* 1 *only.*

31. Given $y^3+z=127$, $y^3+x=135$, and $x^3+y^3+z^3=1133$: to find x, y, and z.

Ans. $x=10$, $y=5$, *and* $z=2$.

32. Given $x^2+xy=108$, $y^2+yz=69$, and $z^2+xz=580$; to find x, y, and z.

Ans. $x=9$, $y=3$, *and* $z=20$.

33. To find two mean proportionals between any two given numbers a and b. *Ans.* $\sqrt[3]{a^2b}$ *and* $\sqrt[3]{b^2a}$.

34. Given $x+yz=384$, $y+xz=237$, and $z+xy=192$; to find x, y, z. *Ans.* $x=10$, $y=17$, *and* $z=22$.

35. To find the least number, which, being divided by 6, 5, 4, 3, and 2, shall leave the remainder 5, 4, 3, 2, and 1, respectively. *Ans.* 59.

36. To find three numbers such, that the sum or difference of any two of them shall be square numbers.

Ans. 1873432, 2399057, *and* 2288168.

37. To find two square numbers such, that their sum may be a square, and their difference a cube, and the side of the said square and cube equal to each other.

Ans. $\frac{784}{15625}$, *and* $\frac{441}{15625}$.

38. To determine the number of fifteens that can be made out of a common pack of 52 cards.

Ans. 17264.

39. Given the rates a and b of two ingredients, and the rate c of the compound m, to find what portions x and y of each must be taken to compose the mixture.

$$Ans.\ x=m\times\frac{c-b}{a-b},\ y=m\times\frac{a-c}{a-b}.$$

40. Given $x^2+xy+y^2=1087$, and $x^4+x^3y^3+y^4=4577295$; to find x and y.

Ans. $x=21$ *and* $y=17$.

41. Given $x+y+z=78$, $x^2+y^2+z^2=2546$, and $xy-xz-yz=527$; to find x, y, and z.

Ans. $x=41$, $y=28$, *and* $z=9$.

42. Given $x+y=152$, and $(x-y)^{\frac{2}{3}}\times(x-y)^{\frac{3}{2}}=8192$; to find x and y.

Ans. $x=108$, *and* $y=44$.

43. To find three numbers such, that if to the square of each the product of the other two be added, the sums shall be squares.

Ans. 73, 9, 328.

44. Let the number of cards in a pack (p) be distributed into any number of heaps (n), by laying as many cards upon the bottom heap as are sufficient

to make up its number q; then, by having the number of cards remaining in the dealer's hand (r) and the number of heaps (n) given, it is required to find the sum of all the bottom cards.

Ans. $(q+1)\times n+(r-p)=$ *sum required.*

45. To find 3 numbers such, that if each be subtracted from the cube of their sum, the remainder shall be cubes. *Ans.* $\frac{11851}{85184}$, $\frac{19467}{85184}$, *and* $\frac{10954}{85184}$.

46. Given the cycle of the sun 18, the golden number 8, and the Roman indiction 10; to find the year.

Ans. 1717.

47. To find 3 cube numbers such, that their sum shall be both a square and a cube number: and if that sum be squared it shall be a cube, and if it be cubed it shall be a square. *Ans.* $\frac{1}{8}a^6$, $\frac{8}{27}a^6$, $\frac{125}{216}a^6$.

48. To find 3 numbers such, that if each be added to the cube of their sum, their sums shall be cubes.

Ans. $\frac{23625}{157464}$, $\frac{1538}{157464}$, $\frac{18577}{157464}$.

49. With guineas and moidores, the fewest, which way,
Three hundred and fifty-one pounds can I pay?
If paid every way 'twill admit of, what sum
Do the pieces amount to?—my fortune's to come.

Ans. 9 *guineas, and* 233 *moidores; and* 37 *ways, which is* $=12987l.$

50. Given $xyzz^{\frac{1}{2}}=zx^{\frac{2}{3}}=100$; to find the value of x and y. *Ans.* $x=47.706$ *and* $z=1.42$.

51. Given $x^{3x}-21x^{2x}+147x^{x}=316$, to find the value of x. *Ans.* $x=2$.

52. Given $44000x^2+1=z^2$; to find x and z in whole numbers.

Ans. $x=40482981221781$ *and* $z=8491781142001$.

53. To find three whole numbers such, that the excess of the greatest above the middle number shall be to the excess of the middle number above the least, as 3 to 1; and also that the sum of every two of these shall be squares. *Ans.* $4n\times1362$, $4n\times402$, *and* $4n\times82$.

54. Given $x+y=a(2)$, and $x^9+y^9=b\,(32)$, to find x and y by quadratics.

Ans. $x=1.4697175$ *and* $y=.5302824$.

55. Given $x^y=500$, and $y^x=300$; to find x and y.

Ans. $x=4.6914$ *and* $y=5.5102$.

56. Given $xy\times(x+z)^2=300$, $xz\times(y+z)^2=1296$, and $yz\times(x+y)^2=432$; to find x, y, and z.

Ans. $x=1$, $y=3$, $z=9$.

57. Given $w^3+x+y+z=57$, $w+x^3+y+z=1763$, $w+x+y^3+z=1350$, and $w+x+y+z^3=153$; to find x, y, z, and w.

Ans. $x=14$, $y=11$, $z=5$, *and* $w=3$.

58. Given $x+y=1750$, $xz+yv=22708$, $xv+yz=12292$, and $xzv+vzy=159252$; to find x, y, z, and v.

Ans. $x=1743$, $y=7$, $z=13$, *and* $v=7$.

59. Given $5x+7y+9z=93256$; to find all the different solutions in affirmative integers which the equation will admit of. *Ans.* 13801148.

60. To find a square number such, that the sum of all its aliquot parts shall be a square number. *Ans.* 2401.

61. To find two square numbers such, that either of them, when added to its aliquot parts, shall make the same sum. *Ans.* 106276 *and* 165649.

62. To find 4 whole nu[redacted]ſuch, that the difference of every two ſhall be a ſq[redacted]mber.

Ans. 1873432, 2288168, 2399057, *and* 6560657.

63. To find three numbers ſuch, that if their ſum be multiplied by the firſt, it ſhall be a triangular number, by the ſecond a ſquare, and by the third a cube.

Ans. $\frac{1}{6}$, $\frac{25}{6}$, *and* $\frac{8}{6}$.

64. To find three biquadrate numbers, the ſum of which will be a ſquare. *Ans.* 12^4, 15^4, *and* 20^4.

65. To find a right-angled triangle ſuch that its perimeter ſhall be a cube, and the perimeter together with the area a ſquare.

Ans. Perp. $=\frac{48406}{2475}$, *base* $=\frac{44352}{2475}$, *hyp.* $=\frac{65648}{2475}$.

66. To find two different iſoſceles triangles ſuch, that their areas and perimeters ſhall be both equal.

Ans. Sides of the one =29, 29, 40.

Ditto of the other =37, 37, 24.

THE END.